

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2410 — Introduction to Complex Analysis

Day of examination: Tuesday 28. November 2023

Examination hours: 09:00–13:00

This problem set consists of 5 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

For each question you may use results from the preceding questions even if you haven't answered them.

Problem 1 (weight 10%)

We consider the complex number z defined by:

$$z = \frac{1}{2} + \frac{\sqrt{3}}{2}i. \quad (1)$$

Provide a geometric interpretation of z and deduce that $z^6 = 1$.

We have $|z|^2 = 1/4 + 3/4 = 1$. Hence $|z - 0| = |z - 1| = |1 - 0| = 1$. The triangle with vertices $0, 1, z$ is equilateral. It follows that $z = \exp(\pi i/3)$. Therefore $z^6 = \exp(6\pi i/3) = \exp(2\pi i) = 1$.

Problem 2 (weight 10%)

Define the function $f : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ by, for $z \neq 0$, $f(z) = 1/z$. Justify that f does not have a primitive on its domain of definition.

Suppose g is a primitive. Consider the path $\gamma : [0, 2\pi] \rightarrow \mathbb{C}$ defined by $\gamma(t) = \exp(it)$.
We would have:

$$2\pi i = \int_{\gamma} f(z) dz = g(\gamma(2\pi)) - g(\gamma(0)) = g(1) - g(1) = 0. \quad (2)$$

Problem 3 (weight 10%)

Let $D \subseteq \mathbb{C}$ be a domain. Let $f : D \rightarrow \mathbb{C}$ be an analytic function. Suppose also that the conjugate $\overline{f}$ is analytic. Prove that f is constant.

(Continued on page 2.)

We decompose f into real and imaginary parts as $f = u + vi$.
The Cauchy-Riemann equations for f are:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad (3)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \quad (4)$$

The Cauchy-Riemann equations for $\bar{f} = u - vi$ are:

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad (5)$$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}. \quad (6)$$

Together these equations give:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0, \quad (7)$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 0. \quad (8)$$

Since D is a domain u and v are both constant, so f is constant.

Problem 4 (weight 20%)

a (weight 10%)

Let the complex function f be defined by:

$$f(z) = \frac{1}{(z^2 + 1)^2}. \quad (9)$$

Determine the poles of f , their orders and the corresponding residues.

(Continued on page 3.)

We have:

$$f(z) = \frac{1}{(z-i)^2(z+i)^2}. \quad (10)$$

Hence f has two poles located at i and $-i$ and of order 2.

$$\operatorname{Res}[f, i] = \lim_{z \rightarrow i} \frac{d}{dz} (z-i)^2 f(z), \quad (11)$$

$$= \lim_{z \rightarrow i} -2(z+i)^{-3}, \quad (12)$$

$$= -i/4. \quad (13)$$

$$\operatorname{Res}[f, -i] = \lim_{z \rightarrow -i} \frac{d}{dz} (z+i)^2 f(z), \quad (14)$$

$$= \lim_{z \rightarrow -i} -2(z-i)^{-3}, \quad (15)$$

$$= i/4. \quad (16)$$

b (weight 10%)

Compute the integral:

$$\int_{-\infty}^{+\infty} f(z) dz. \quad (17)$$

Let $C(R)$ be the semicircle in the upper half plane connecting R to $-R$.

We have:

$$\int_{-R}^R f(z) dz + \int_{C(R)} f(z) dz = 2\pi i \operatorname{Res}[f, i] = \pi/2. \quad (18)$$

Also for $R > 1$ we have, by the ML estimate:

$$\left| \int_{C(R)} f(z) dz \right| \leq \pi R \frac{1}{(R^2 - 1)^2} \rightarrow 0. \quad (19)$$

This lets us conclude.

We get:

$$\int_{-\infty}^{+\infty} f(z) dz = \pi/2. \quad (20)$$

Problem 5 (weight 10%)

Let P be the polynomial defined by $P(z) = z^5 + 3z - 1$. Justify that P has one and only one root in the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$.

(Continued on page 4.)

For $|z| = 1$ we have $|z^5| = 1$ and $|3z - 1| \geq 3|z| - 1 = 2$. By Rouché's theorem P has as many roots as $3z - 1$ in D (counting multiplicities). The latter polynomial has exactly one root ($1/3$).

Problem 6 (weight 20%)

We define a function $u : \mathbb{C} \rightarrow \mathbb{R}$ by, for $x, y \in \mathbb{R}$:

$$u(x + iy) = e^{-x} \sin(y). \quad (21)$$

a (weight 10%)

Justify that u has a harmonic conjugate and compute one.

One checks that u is harmonic by a direct calculation. Since also $\mathbb{C}$ is starshaped it follows that u has a harmonic conjugate.

A harmonic conjugate v is determined by:

$$v(x', y') = \int \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy, \quad (22)$$

$$= \int -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy, \quad (23)$$

$$= - \int e^{-x} \cos(y) dx + e^{-x} \sin(y) dy, \quad (24)$$

$$= - \int_{(0,0)}^{(x',0)} e^{-x} dx - \int_{(x',0)}^{(x',y')} e^{-x} \sin(y) dy, \quad (25)$$

$$= e^{-x} - 1 + e^{-x} (\cos(y) - 1), \quad (26)$$

$$= e^{-x} \cos(y) - 1. \quad (27)$$

By the way :

$$e^{-x} \sin(y) + e^{-x} \cos(y)i = e^{-x} (\cos(-y) + i \sin(-y))i, \quad (28)$$

$$= \exp(-(x + iy))i. \quad (29)$$

b (weight 10%)

What is the average of u on the circle:

$$\partial B(i\pi/2, \pi/2) = \{z \in \mathbb{C} : |z - i\pi/2| = \pi/2\} \quad (30)$$

Since u is harmonic, the average is the value at $i\pi/2$, which is 1.

Problem 7 (weight 20%)

For $z \in \mathbb{C} \setminus \{1\}$ we define:

$$f(z) = \exp\left(\frac{1}{1-z}\right). \quad (31)$$

(Continued on page 5.)

We also introduce the real part of f as the function u defined by:

$$u(z) = \Re f(z). \quad (32)$$

a (weight 10%)

We let D be the open unit disc:

$$D = \{z \in \mathbb{C} : |z| < 1\}. \quad (33)$$

Justify that u is harmonic on D and that u is also unbounded on D .

Since f is the composition of a rational function and the exponential, f is analytic. Therefore u is harmonic.

We have:

$$\lim_{z \rightarrow 1^-} u(z) = +\infty. \quad (34)$$

Hence u is unbounded on D .

b (weight 10%)

We let C be unit circle:

$$C = \{z \in \mathbb{C} : |z| = 1\}. \quad (35)$$

Check that for $z \in C \setminus \{1\}$ we have $|f(z)| = \exp(1/2)$. In particular u is bounded on $C \setminus \{1\}$ but not on D . Why is there no contradiction with the maximum principle?

We have for $z = \exp(i\theta) \in C$:

$$f(z) = \exp\left(\frac{1}{1 - e^{i\theta}}\right) = \exp\left(\frac{-e^{-i\theta/2}}{e^{i\theta/2} - e^{-i\theta/2}}\right), \quad (36)$$

$$= \exp(-1/(2i)(\cos(i\theta/2) - i \sin(\theta/2))/\sin(\theta/2)), \quad (37)$$

$$= \exp(1/2 + i/2 \tan(\theta/2)^{-1}). \quad (38)$$

This gives $|f(z)| = \exp(1/2)$.

There is no contradiction with the maximum principle, since it requires u to be continuous on $\overline{D}$, in particular at 1.

THE END