

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT 2410 — Introduction to Complex Analysis

Day of examination: Monday, 2 December 2024

Examination hours: 15.00 – 19.00

This problem set consists of 7 pages.

Appendices: None

Permitted aids: none.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: The exam consists of 10 sub-problems, each giving maximally 10 points. To get a point for a sub-problem it is expected that you give an explanation of your solution. Partial scores for partially correct arguments will be given, so try to do all the best you can. Note that each problem is independent of other problems, unless it is otherwise explicitly stated.

Good Luck!

(Continued on page 2.)

Problem 1

Let $g(z) = z^2(z - 2)$.

1a (weight 10 points)

Let $0 < r < 1$ and define $D = \{z \in \mathbf{C} : |z - 1| < r\}$. Show that there is an analytic function $f : D \rightarrow \mathbf{C}$ so that $f(z)^3 = g(z)$ for all $z \in D$.

Hint: Since the domain is simply connected, and the function g has no roots inside - and even on the boundary - of the domain, the existence of such a holomorphic function $f(z)$ follows from the lectures in the course/book.

1b (weight 10 points)

Let $f(z)$ be an analytic function as in sub-problem 1a, and define $\zeta = f(1)$. Show that $\zeta^3 = -1$, and compute $f'(1)$ as a function of ζ .

Hint: Substitute $z = 1$ into the equation $f(z)^3 = z^2(z - 2)$, we obtain $f(1)^3 = -1$. Since $\zeta = f(1)$, we obtain $\zeta^3 = -1$. Now we differentiate both sides of the equation $f(z)^3 = z^2(z - 2)$, and obtain $3f(z)^2 f'(z) = 2z(z - 2) + z^2$. Substituting $z = 1$ into this, we get: $3f(1)^2 f'(1) = -1$. Hence, since $\zeta = f(1)$, we obtain $3\zeta^2 f'(1) = -1$, and hence $f'(1) = -\frac{1}{3\zeta^2}$.

Problem 2 (weight 10 points)

Let

$$f(z) = \frac{-z + 1}{2z + 3}.$$

Compute the inverse function to $f(z)$ (i.e. a function $g(z)$ so that $g(f(z)) = z$ identically), and find the image under $f(z)$ of the circle $\{z \in \mathbf{C} : |z - 1| = 2\}$

Hint: Define

$$w = \frac{-z + 1}{2z + 3}.$$

Solving z in terms of w we obtain:

$$z = \frac{-3w + 1}{2w + 1}.$$

Therefore, the image of the circle $\{z \in \mathbf{C} : |z - 1| = 2\}$ is the set:

$$\left| \frac{-3w + 1}{2w + 1} - 1 \right| = 1.$$

The latter can be simplified to $|5w| = |2w + 1|$. If we write $w = u + iv$, with $u, v \in \mathbf{R}$, and square both sides of the last equation, then we get: $25(u^2 + v^2) = (2u + 1)^2 + (2v)^2$. Bring to the same side, and complete the squares (separately in each variable u and v), we see that the image is also a circle.

(Continued on page 3.)

Problem 3

Let

$$g(z) = \frac{\cos(z) - 1}{z^2 \sin(z)}.$$

3a (weight 10 points)

Determine all the poles w of $g(z)$ which satisfy $|w| \leq 2\pi$, and the order at each of these poles.

Hint: Since the numerator and denominator, $\cos(z) - 1$ and $z^2 \sin(z)$, are entire functions, if w is a pole of $g(z)$, it must be a root of $z^2 \sin(z)$. In the domain $|w| \leq 2\pi$, the function $z^2 \sin(z)$ has 5 roots: $0, \pi, 2\pi, -\pi, -2\pi$. We will explore each of these points to determine if it is a pole, and if so, what the order is. First, consider the point $z = 0$. It is a root of order 2 of z^2 , and a root of order 1 of $\sin(z)$ (which can be checked by checking that $\sin(z) = 0$ at $z = 0$, while $(\sin(z))' = \cos(z)$ is nonzero at $z = 0$). Hence, $z = 0$ is a root of order 3 of the denominator. Now, we check if it is a root of the numerator, and if so, of what order. Indeed, $z = 0$ is a root of order 2 of the numerator (which can be seen that both $\cos(z) - 1$ and its derivative $-\sin(z)$ are 0 at $z = 0$, while its second derivative $-\cos(z)$ is nonzero at $z = 0$). Since $2 < 3$, the point $z = 0$ is a pole of the function $g(z)$, of order $3 - 2 = 1$. We now consider the point $z = \pi$. Similar to the case $z = 0$, we can see that the denominator has a root of order 1 at π , while the numerator does not have a root there. Hence, π is a pole of $g(z)$ of order 1. We now consider the point $z = 2\pi$. Here the point 2π is a root of both numerator and denominator, and of order 1 for the denominator. Hence, 2π is not a pole of $g(z)$. The point $-\pi$ is similar to the point π , and the point -2π is similar to the point 2π .

3b (weight 10 points)

Let

$$g(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

be the Laurent series of $g(z)$ about $z_0 = 0$. Determine a_{-3} , a_{-2} , a_{-1} and a_0 .

Hint: Since 0 is a pole of order 1 of $g(z)$, the Laurent series of $g(z)$ about 0 has the form:

$$g(z) = \frac{a_{-1}}{z} + a_0 + a_1 z + a_2 z^2 + \dots$$

In particular, $a_{-3} = a_{-2} = 0$. To find a_1 and a_0 , we define

$$h(z) = zg(z) = a_{-1} + a_0 z + a_1 z^2 + a_2 z^3 + \dots,$$

and observe that $a_{-1} = h(0)$, $a_0 = h'(0)$. Using the Taylor's series for $\cos(z) - 1$ and $\sin(z) = 1$ about 0, we find:

$$h(z) = zg(z) = \frac{-z^2/2! + z^4/4! - z^6/6! + \dots}{z^2 - z^4/3! + z^6/5! + \dots}$$

(Continued on page 4.)

After cancelling z^2 from both numerator denominator, we have

$$h(z) = \frac{-1/2! + z^2/4! - z^4/6! + \dots}{1 - z^2/3! + z^4/5! + \dots}$$

From this we find: $a_{-1} = h(0) = -1/2$, $a_0 = h'(0) = 0$.

Problem 4

Let $f(z)$ be an entire function. Recall that $\mathbf{Z}$ is the set of integer numbers. Assume that for all $z \in \mathbf{C} \setminus \pi\mathbf{Z}$, we have:

$$\frac{|f(z)|}{|\sin(z)|} \leq 2024(|z| + 1).$$

4a (weight 10 points)

Define $g : \mathbf{C} \setminus \pi\mathbf{Z} \rightarrow \mathbf{C}$ by the formula: $g(z) = f(z)/\sin(z)$. Show that $g(z)$ is analytic on $\mathbf{C} \setminus \pi\mathbf{Z}$, and can be extended to an entire function $G(z)$.

Hint: Since both numerator and denominator of $g(z)$ are entire function, $g(z)$ is automatically analytic at all points where the denominator is nonzero. Hence $g(z)$ is analytic on $\mathbf{C} \setminus \pi\mathbf{Z}$. At each point in $\pi\mathbf{Z}$, the function $g(z)$ is bounded by the inequality in the question, hence by Riemann's removable singularity theorem, the function $g(z)$ extends analytically at that point. Therefore, $g(z)$ extends to an entire function $G(z)$.

4b (weight 10 points)

Show that there are constants a and b such that $f(z) = (az + b)\sin(z)$ for all $z \in \mathbf{C}$. *Hint:* You can consider first the function $G(z)$ in sub-problem 4a.

Hint: Note that $f(z) = \sin(z)G(z)$ on $\mathbf{C} \setminus \pi\mathbf{Z}$, and both sides of the equality are entire functions. Therefore, by the identity theorem for analytic functions, $f(z) = G(z)\sin(z)$ on $\mathbf{C}$. Now we show that there are $a, b \in \mathbf{C}$ so that $G(z) = az + b$ for all z . This indeed follows from the generalisation of Liouville's theorem from the exercises in the textbook, and also was discussed in the lectures.

Problem 5

Define

$$f(z) = \frac{e^{-iz}}{(z+i)^2(z^2+4)}.$$

5a (weight 10 points)

Determine all singular points of f and compute the residue of f at those points.

(Continued on page 5.)

Hint: Since the numerator and denominator are entire function, and the numerator has no root, a point is a singular point of $f(z)$ if and only if it is a root of the denominator $(z+i)^2(z^2+4)$. Therefore, the singular points of $f(z)$ are $-i$ (order 2), $2i$ (order 1) and $-2i$ (order 1). Now we compute the residue at these points. We first look at the point $2i$, which is a pole of order 1. The Laurent series about $2i$ then has the form

$$f(z) = \frac{a_{-1}}{z-2i} + a_0 + a_1(z-2i) + a_2(z-2i)^2 + \dots$$

Multiply both sides with $z-2i$, we find:

$$\frac{e^{-iz}}{(z+i)^2(z+2i)} = a_{-1} + a_0(z-2i) + a_1(z-2i)^2 + \dots$$

Therefore,

$$\text{Res}[f(z), 2i] = a_{-1} = \frac{e^{-iz}}{(z+i)^2(z+2i)} \Big|_{z=2i} = \frac{e^2}{36}i.$$

Similarly,

$$\text{Res}[f(z), -2i] = \frac{e^{-iz}}{(z+i)^2(z-2i)} \Big|_{z=-2i} = -\frac{e^{-2}}{4}i.$$

Next, we compute the residue at $-i$, which is a pole of order 2. The Laurent series of $f(z)$ about $-i$ is:

$$f(z) = \frac{a_{-2}}{(z+i)^2} + \frac{a_{-1}}{z+i} + a_0 + a_1(z+i) + a_2(z+i)^2 + \dots$$

Define

$$h(z) = (z+i)^2 f(z) = \frac{e^{-iz}}{z^2+4},$$

we have

$$h(z) = a_{-2} + a_{-1}(z+i) + a_0(z+i)^2 + a_1(z+i)^3 + \dots$$

is an analytic function about $-i$. Therefore, $\text{Res}[f(z), -i] = h'(-i)$. We rewrite $h(z) = e^{-iz}(z^2+4)^{-1}$, and compute $h'(z) = -ie^{-iz}(z^2+4)^{-1} - 2ze^{-iz}(z^2+4)^{-2}$. From this we find

$$\text{Res}[f(z), -i] = -\frac{e^{-1}}{9}i.$$

5b (weight 10 points)

Show that if $z = x + iy$ with $y \leq 0$, then $|e^{-iz}| \leq 1$. With the help of this fact, compute

$$\int_{-\infty}^{\infty} f(x) dx.$$

(Continued on page 6.)

Hint: If $z = x + iy$ with $y \leq 0$, then $|e^{-iz}| = |e^{y-ix}| = e^y \leq 1$. Now, for each $R > 2$, consider D_R the lower half of the disc with center 0 and radius R , i.e. $D_R = \{z : z = re^{i\theta}, \theta \in [\pi, 2\pi], 0 < r < R\}$. The disc D_R contains all poles in the lower halfplane of $f(z)$, which are $-i$ and $-2i$. The boundary of D_R has 2 components: the real interval $[-R, R]$ and the lower half circle $U_R = \{z : z = Re^{i\theta}, \theta \in [0, 2\pi]\}$. The positive orientation is as follows: it goes on the real interval from R to $-R$, then goes on the lower half circle from $-R$ to R . By Residue's theorem, we have

$$-\int_{-R}^R f(x)dx + \int_{\pi}^{2\pi} f(z)dz = 2\pi i(\text{Res}[f(z), -i] + \text{Res}[f(z), -2i]).$$

For the first term, we have:

$$-\int_{-R}^R f(x)dx \rightarrow -\int_{-\infty}^{\infty} f(x)dx,$$

as $R \rightarrow \infty$. Now we consider the second term, i.e. the integral on the lower half circle. First, for $z = Re^{i\theta} = (x, y)$ with $\theta \in [\pi, 2\pi]$ we have

$$|f(z)| = \frac{|e^{-iz}|}{|(z+i)^2(z^2+4)|} \leq \frac{1}{(R-1)^2(R^2-4)}.$$

Since the length of the half circle is πR , we have by ML-estimate

$$|\int_{\pi}^{2\pi} f(z)dz| \leq \frac{1}{(R-1)^2(R^2-4)}\pi R,$$

which converges to 0 as $R \rightarrow \infty$. Therefore,

$$-\int_{-\infty}^{\infty} f(x)dx = 2\pi i[-\frac{e^{-1}}{9}i - \frac{e^{-2}}{4}i].$$

Multiplying both sides with -1 , we obtain the final result:

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i[\frac{e^{-1}}{9}i + \frac{e^{-2}}{4}i] = -2\pi[\frac{e^{-1}}{9} + \frac{e^{-2}}{4}].$$

Problem 6 (weight 10 points)

Let $P(z) = z^8 + 13z^5 + 17z^3 + 2$. Show that $P(z)$ has precisely (counting multiplicities):

3 roots in the set $\{z \in \mathbf{C} : |z| < 1\}$, 2 roots in the set $\{z \in \mathbf{C} : 1 < |z| < 2\}$, and 3 roots in the set $\{z \in \mathbf{C} : 2 < |z|\}$.

Hint: On the set $\{z \in \mathbf{C} : |z| < 1\}$ we apply Rouché theorem for $f_1(z) = 17z^3$, $g_1(z) = z^8 + 13z^5 + 2$. We have $f_1(z) + g_1(z) = P(z)$, and on $\{|z| = 1\}$ we have

$$|f_1(z)| = 17 > 16 = |z^8| + |13z^5| + |2| \geq |g_1(z)|.$$

(Continued on page 7.)

Therefore, in $\{|z| < 1\}$, $P(z)$ has the same number of roots as $f_1(z)$, which is 3. On the set $\{|z| < 2\}$, we apply Rouché theorem for $f_2(z) = 13z^5$ and $g_2(z) = z^8 + 17z^3 + 2$, which satisfy $f_2(z) + g_2(z) = P(z)$, and which on $\{|z| = 2\}$ satisfy

$$|f(z)| = 416 > 394 = |z^8| + |17z^3| + |2| \geq |g_2(z)|.$$

Therefore, on the set $\{|z| < 2\}$, $P(z)$ has the same number of roots as $f_2(z)$, which is 5. Since $P(z)$ has no root on $\{|z| = 1\}$, it follows that $P(z)$ has $5 - 3 = 2$ roots in $\{1 < |z| < 2\}$. Since $P(z)$ is a polynomial of degree 8, it has 8 roots in $\mathbf{C}$. Since $P(z)$ has no root on $\{|z| = 2\}$, it follows that $P(z)$ has $8 - 5 = 3$ roots in $\{2 < |z|\}$.