

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT2410 — Introduction to Complex Analysis

Day of examination: Monday, December 15, 2025

Examination hours: 09:00–13:00

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

1a

Show that the function

$$v(z) = \log(x^2 + y^2), \quad z = x + iy,$$

is harmonic in $D = \mathbb{C} \setminus \{0\}$.

Possible solution: Calculate

$$\begin{aligned} \frac{\partial v}{\partial x} &= \frac{2x}{x^2 + y^2}, & \frac{\partial v}{\partial y} &= \frac{2y}{x^2 + y^2} \\ \frac{\partial^2 v}{\partial x^2} &= \frac{2y^2 - 2x^2}{(x^2 + y^2)^2}, & \frac{\partial^2 v}{\partial y^2} &= -\frac{2y^2 - 2x^2}{(x^2 + y^2)^2}. \end{aligned}$$

1b

Prove that there does not exist a function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, such that $f := u + iv$ is analytic in $\mathbb{C} \setminus \{0\}$. (**Hint:** recall that $\arctan'(u) = 1/(1 + u^2)$ and that $\arctan(a) + \arctan(1/a) = \text{sign}(a)\pi/2$ if $a \neq 0$.)

Possible solution: Assume that u exists. Then

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = -\frac{2x}{x^2 + y^2}, \end{aligned}$$

which implies that

$$u(x, y) = 2 \arctan(x/y) + c_1(y) = -2 \arctan(y/x) + c_2(x).$$

(Continued on page 2.)

Using the equality $\arctan(a) + \arctan(1/a) = \text{sign}(a)\pi/2$,

$$c_2(y) = \text{sign}(x/y)\pi + c_1(x).$$

Thus either c_1 or c_2 must be discontinuous.

Problem 2

Prove that the function $f(z) = \cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$ maps the lines $\text{Re}(z) = \text{const.}$ on hyperbolas, and the lines $\text{Im}(z) = \text{const.}$ on ellipses.

Possible solution: We have that

$$\begin{aligned} f(z) &= u + iv \\ &= \frac{1}{2}(e^{ix-y} + e^{-ix-y}) \\ &= \frac{1}{2}(e^y + e^{-y})\cos(x) - \frac{i}{2}(e^y - e^{-y})\sin(x). \end{aligned}$$

So if y is constant

$$\frac{u^2}{\cosh^2(y)} + \frac{v^2}{\sinh^2(y)} = 1 \quad \longleftarrow \quad \text{ellipse}$$

and if x is constant

$$\frac{u^2}{\cos^2(x)} - \frac{v^2}{\sin^2(x)} = 1 \quad \longleftarrow \quad \text{hyperbola}$$

Problem 3

Let

$$h(z) = z + \frac{1}{z}.$$

3a

Find all zeros and singularities of h and calculate the integral

$$\int_{|z|=2} \frac{1 - 1/z^2}{z + 1/z} dz,$$

where the contour is traversed in the positive direction. (**Hint:** Use the argument principle.)

Possible solution: h has a pole of order 1 at $z = 0$ and simple zeros at $z = \pm i$. We recognise the integrand as h'/h and by the argument principle the integral equals $2\pi i(2 - 1)$.

(Continued on page 3.)

3b

Find two functions $z_{\pm}(w)$ such that

- (i) $z_{\pm}$ are analytic in $\mathbb{C} \setminus \{[-2, 2]\}$,
- (ii) $h(z_{\pm}(w)) = w$ for $w \in \mathbb{C}$.

(**Hint:** Use the principal branch of the complex square root. Consider where the branch is analytic.)

Possible solution: We solve the equation $h(z) = w$ to get

$$z_{\pm}(w) = \frac{1}{2} \left(w \pm \sqrt{w^2 - 4} \right)$$

If we use the principal branch of the square root, which is analytic when its argument is not real and negative, we see that the last expression is analytic in $\mathbb{C} \setminus \{[-2, 2]\}$ since

$$w^2 - 4 \text{ is real and negative} \Leftrightarrow w \in [-2, 2].$$

Problem 4

Let f be an entire function and suppose that there exists positive (real) constants R and C such that

$$|z| > R \implies |f(z)| \leq C |z|^n,$$

for some positive integer n .

4a

Show that for any ξ with $|\xi| > 2R$

$$f^{(m)}(\xi) = 0 \text{ for } m > n.$$

(**Hint:** Use Cauchy's integral formula for the disk $C_R(\xi) = \{|z - \xi| = R\}$ and make an upper estimate for the integrand.)

Possible solution: Since $|\xi| > 2R$ we have that if z is on the circle $\{|z - \xi| = R\}$ then $|z| > R$. By Cauchy's formula for derivatives

$$\begin{aligned} \left| f^{(m)}(\xi) \right| &= \frac{m!}{2\pi} \left| \int_{C_R(\xi)} \frac{f(z)}{(z - \xi)^{m+1}} dz \right| \\ &\leq \frac{m!C}{2\pi} \int_{C_R(\xi)} \frac{(R + 2R)^n}{|z - \xi|^{m+1}} dz \\ &\leq \frac{m!C}{2\pi} \frac{(3R)^n 2\pi R}{R^{m+1}} \rightarrow 0 \text{ as } R \rightarrow \infty. \end{aligned}$$

(Continued on page 4.)

4b

Show that f is a polynomial of degree less than or equal to n .

Possible solution: Since f is entire, its Taylor expansion around any z_0 converges, in particular for $|z_0| > 2R$. Thus

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(z_0) (z - z_0)^k \\ &= \sum_{k=0}^n \frac{1}{k!} f^{(k)}(z_0) (z - z_0)^k, \end{aligned}$$

which is a polynomial of degree less than or equal to n .

Problem 5

Let $f(z) = z^3 + 4z^2 + 2$. Determine the number of zeros of f in the closed ball $\{|z| \leq 1\}$ and show that they are simple.

Possible solution: Set $F = 4z^2 + 2$ and $G = z^3$. On the unit circle we have

$$|F(z)| \geq 2 > 1 = |G(z)|.$$

Inside the circle, F has two roots, $\pm i/\sqrt{2}$. By Rouché's theorem f has two roots in $\{|z| \leq 1\}$. $f'(z) = 3z^2 + 8z = z(3z + 8)$ has zeros 0 and $-8/3$, none of which are zeros of f . Hence the zeros of f are simple.

Problem 6

6a

Find the singularities and the corresponding residues, of the function

$$f(z) = \frac{z^\alpha}{1 + z^2},$$

where α is a real number in the open interval $(0, 1)$.

Possible solution: f has simple poles at $\pm i$, with residues

$$\operatorname{Res}(f, i) = \frac{i^\alpha}{2i}, \quad \operatorname{Res}(f, -i) = -\frac{(-i)^\alpha}{2i}$$

6b

Determine the improper integral

$$I = \int_0^\infty f(x) dx.$$

(Continued on page 5.)

(**Hint:** Use a branch cut of the logarithm along the positive real axis and a contour of the type shown in Figure 1.)

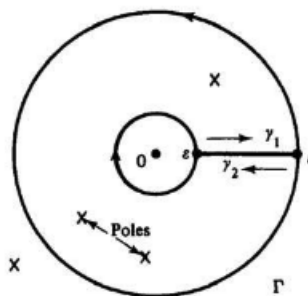


Figure 1: A keyhole contour for Problem 6b.

Possible solution: We use a branch of the logarithm with a branch cut along the positive real axis, so that $0 \leq \arg(z) < 2\pi$. Let C_R denote the circle $\{|z| = R > 1\}$ traversed in the positive direction, γ_- the line from R to $r < 1$ “below” the real axis, C_r the circle of radius r and center 0, traversed in the negative direction and γ_+ the straight line from r to R “above” the real axis. Set $\Gamma = C_R + \gamma_- + C_r + \gamma_+$. We have

$$\begin{aligned} 2\pi i (\operatorname{Res}(f, i) + \operatorname{Res}(f, -i)) &= \int_{\Gamma} f(z) dz \\ &= \left(\int_{\gamma_+} + \int_{\gamma_-} + \int_{C_R} + \int_{C_r} \right) f(z) dz. \end{aligned}$$

The integrals over C_r and C_R vanish as $r \rightarrow 0$ and $R \rightarrow \infty$ respectively. We have that

$$I = \lim_{\substack{r \rightarrow 0 \\ R \rightarrow \infty}} \int_{\gamma_+} f(z) dz.$$

On γ_-

$$z^\alpha = x^\alpha e^{2\pi\alpha i},$$

so that

$$\lim_{\substack{r \rightarrow 0 \\ R \rightarrow \infty}} \int_{\gamma_-} f(z) dz = -e^{2\pi\alpha i} I.$$

We get

$$\begin{aligned} I(1 - e^{2\pi\alpha i}) &= 2\pi i (\operatorname{Res}(f, i) + \operatorname{Res}(f, -i)) \\ -Ie^{\pi\alpha i} (e^{\pi\alpha i} - e^{-\pi\alpha i}) &= 2\pi i \frac{e^{\frac{\pi}{2}\alpha i} - e^{\frac{3\pi}{2}\alpha i}}{2i} \\ I(e^{\pi\alpha i} - e^{-\pi\alpha i}) &= \pi (e^{\frac{\pi}{2}\alpha i} - e^{-\frac{\pi}{2}\alpha i}) \\ I \sin(\alpha\pi) &= \pi \sin\left(\alpha \frac{\pi}{2}\right), \end{aligned}$$

so that

$$I = \pi \frac{\sin(\alpha \frac{\pi}{2})}{\sin(\alpha\pi)} = \frac{\pi}{2} \frac{1}{\cos(\alpha \frac{\pi}{2})}.$$

(Continued on page 6.)

THE END