

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Deferred exam in: MAT2410 — Introduction to Complex Analysis

Day of examination: Wednesday, January 21, 2026

Examination hours: 15:00–19:00

This problem set consists of 7 pages.

Appendices: None.

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Assume that $a \in (0, 1)$ and let f be the function

$$f(z) = \frac{e^{az}}{1 + e^z}.$$

1a

Find all zeros and poles of f .

Possible solution: f is never zero, and has poles when $e^z = -1$, i.e., $z = (2k + 1)\pi i$ with $k \in \mathbb{Z}$.

1b

Show that

$$I = \int_{-\infty}^{\infty} f(x) dx = \frac{\pi}{\sin(\pi a)}.$$

(**Hint:** Use the positively oriented rectangular contour with vertices at $-R, R, R + 2\pi i, -R + 2\pi i$.)

Possible solution: We name the whole contour Γ and its four parts $\gamma_{1,2,3,4}$. There is one pole inside Γ ; $z = \pi i$,

$$\operatorname{Res}(f, \pi i) = \lim_{z \rightarrow \pi i} (z - \pi i) f(z) = -e^{\pi a i}.$$

Therefore $\int_{\Gamma} f(z) dz = -2\pi i e^{\pi a i}$. We have that a should be both exponents

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of numerator, not in denom.?

$$\begin{aligned} \left| \int_{\gamma_2} f(z) dz \right| &= \left| \int_0^{2\pi} \frac{e^{ti} e^{aR}}{1 + e^{ati} e^R} i dt \right| \\ &\leq 2\pi \frac{e^{aR}}{e^R - 1} \rightarrow 0 \text{ as } R \rightarrow \infty. \end{aligned}$$

Similarly $\lim_{R \rightarrow \infty} \int_{\gamma_4} = 0$. We also see that

$$\int_{\gamma_3} f(z) dz = -e^{2\pi ai} \int_{\gamma_1} f(z) dz.$$

Sending R to ∞ we get

$$(1 - e^{2\pi ai}) I = -2\pi i e^{\pi ai},$$

and dividing by $-2ie^{\pi ai}$

$$\sin(\pi a) I = \pi.$$

Problem 2

Let now

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

2a

Show that this improper integral on the right hand side above converges absolutely for $\operatorname{Re}(z) > 1$ and thus Γ is well defined here. Prove that Γ is analytic for $\operatorname{Re}(z) > 1$. (**Hint:** for the first part, split the integral into the two intervals $t \in [0, 1]$ and $t \in (1, \infty)$.)

Possible solution: Write $a = \operatorname{Re}(z)$. For $t > 0$ we have

$$|t^{z-1} e^{-t}| = t^{a-1} e^{-t}.$$

Split the integral at 1:

$$\int_0^\infty |t^{z-1} e^{-t}| dt = \int_0^1 t^{a-1} e^{-t} dt + \int_1^\infty t^{a-1} e^{-t} dt.$$

For $0 < t \leq 1$ we have $e^{-t} \leq 1$, hence

$$\int_0^1 t^{a-1} e^{-t} dt \leq \int_0^1 t^{a-1} dt,$$

and the right-hand side is finite precisely when $a > 0$. Since $\operatorname{Re}(z) = a > 1$, this part converges.

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For $t \geq 1$ the exponential dominates the polynomial factor. For example, the continuous function $t \mapsto t^{a-1}e^{-t/2}$ on $[1, \infty)$ tends to 0 as $t \rightarrow \infty$, so it attains a finite maximum; therefore there exists $C > 0$ such that

$$t^{a-1} \leq Ce^{t/2} \quad \text{for all } t \geq 1,$$

and hence

$$t^{a-1}e^{-t} \leq Ce^{-t/2} \quad \text{for } t \geq 1.$$

Since $\int_1^\infty e^{-t/2} dt < \infty$, the comparison test implies that $\int_1^\infty t^{a-1}e^{-t} dt$ converges.

Combining the two parts shows that

$$\int_0^\infty |t^{z-1}e^{-t}| dt < \infty,$$

so the integral $\int_0^\infty t^{z-1}e^{-t} dt$ converges absolutely when $\operatorname{Re}(z) > 1$.

Since the integral converges absolutely, we can differentiate under the integral to get

$$\Gamma'(z) = \int_0^\infty t^{z-1} \log(t) e^{-t} dt.$$

2b

For $\operatorname{Re}(z) \leq 1$ we define $\Gamma(z)$ by analytic continuation. Show that $\Gamma(1) = 1$, $\Gamma(z+1) = z\Gamma(z)$ and conclude that $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. (**Hint:** integrate by parts.)

Possible solution: We integrate by parts [hint for this?](#)

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt = z \int_0^\infty t^{z-1} e^{-t} dt = z\Gamma(z).$$

$$\Gamma(1) = \int_0^\infty e^{-t} dt = 1.$$

The factorial formula follows by induction.

2c

Show that

$$\Gamma(z) = \frac{\Gamma(z+n+1)}{(z+n)(z+n-1)(z+n-2)\cdots(z+1)z}, \quad \text{for } n = 0, 1, 2, 3, \dots,$$

and infer that Γ has a simple pole at $z = -n$. Find the residue $\operatorname{Res}(\Gamma, -n)$.

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Possible solution: By applying the recursion formula

$$\begin{aligned}\Gamma(z+n+1) &= (z+n)\Gamma(z+n) \\ &= (z+n)(z+n-1)\Gamma(z+n-1) \\ &\vdots \\ &= (z+n)(z+n-1)(z+n-2)\cdots(z+1)z\Gamma(z).\end{aligned}$$

From the formula we see that

$$\begin{aligned}\operatorname{Res}(\Gamma, -n) &= \lim_{z \rightarrow -n} (z+n)\Gamma(z) \\ &= \lim_{z \rightarrow -n} \frac{\Gamma(z+n+1)}{(z+n-1)(z+n-2)\cdots(z+1)z} \\ &= \frac{\Gamma(1)}{(-1)(-2)\cdots(-(n-1))(-n)} \\ &= \frac{(-1)^n}{n!}.\end{aligned}$$

Problem 3

Let

$$f(z) = \sqrt{z} + \frac{1}{\sqrt{z}},$$

where we use the principal branch of $\sqrt{\cdot}$ so that f is analytic in $D = \mathbb{C} \setminus \{(-\infty, 0]\}$.

3a

Let $C = \{|z| = 4\} \setminus \{-4\}$. Sketch $f(C)$. Recall that an ellipse centered at the origin is given by points (u, v) such that

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} = 1,$$

and that the points satisfying

$$\frac{u^2}{a^2} - \frac{v^2}{b^2} = 1$$

describe an hyperbola.

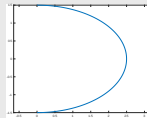
Possible solution: We parametrize C by $z = 4e^{it}$ $t \in (-\pi, \pi)$. Let $w = u + iv = f(z)$, and we get

$$\begin{aligned}w &= 2e^{it/2} + \frac{1}{2}e^{-it/2} & \tau &= t/2 \\ &= \frac{5}{2}\cos(\tau) + i\frac{3}{2}\sin(\tau) = u + iv.\end{aligned}$$

(Continued on page 5.)

Hence $f(C)$ is the right half of the ellipse given by

$$\frac{4}{25}u^2 + \frac{4}{9}v^2 = 1$$



3b

Calculate $\int_C f(z) dz$.

Possible solution: We calculate

$$\begin{aligned} \int_C f(z) dz &= \int_{-\pi}^{\pi} \left(2e^{it/2} + \frac{1}{2}e^{-it/2} \right) ie^{it} dt \\ &= \int_{-\pi}^{\pi} \left(2e^{it3/2} + \frac{1}{2}e^{it/2} \right) i dt \\ &= \frac{4i}{3i} e^{it3/2} \Big|_{-\pi}^{\pi} + \frac{2i}{2i} e^{it/2} \Big|_{-\pi}^{\pi} \\ &= \frac{4}{3} \left(e^{i\pi3/2} - e^{-i\pi3/2} \right) + e^{i\pi/2} - e^{-i\pi/2} \\ &= \frac{4}{3}(-2i) + 2i = -\frac{2}{3}i. \end{aligned}$$

Problem 4

4a

Let f and g be complex numbers different from zero. Suppose that

$$|f + g| = |f| + |g|.$$

Explain why $g = rf$ for a real number $r > 0$.

Possible solution: This is the “=” case of the triangle inequality. It holds iff. f and g are parallel and point in the same direction, i.e. $f = rg$ with r real and positive.

4b

Let D be a bounded simply connected domain with boundary ∂D . Suppose f and g are continuous on the closure of D , analytic in D and have no zeros on ∂D . Suppose $|f(z) + g(z)| = |f(z)| + |g(z)|$ for all $z \in \partial D$. Show that f and g have the same number of zeros in D . (**Hint:** Use the argument principle on $r = f/g$.)

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Possible solution: Define $r(z) = f(z)/g(z)$, then r is continuous on the closure of D and analytic in D , except at the zeros of g . Furthermore, $r(z)$ is real and positive **constant** on ∂D , thus $\text{Arg}(r(z)) = 0$ on ∂D . Hence by the argument principle

$$\begin{aligned} 0 &= \Delta_{\partial D} \text{Arg}(r(z)) = \int_{\partial D} \frac{r'(z)}{r(z)} dz \\ &= N_0(r) - N_p(r) = N_0(f) - N_0(g). \end{aligned}$$

Problem 5

Show that every Möbius transformation

$$T(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0,$$

has at least one fixpoint in the extended complex plane $\mathbb{C} \cup \{\infty\}$.

Possible solution: Let

$$T(z) = \frac{az + b}{cz + d}, \quad ad - bc \neq 0.$$

A fixed point satisfies $T(z) = z$, i.e.,

$$z = \frac{az + b}{cz + d}.$$

Rearranging gives

$$z(cz + d) = az + b \iff cz^2 + (d - a)z - b = 0.$$

Case 1: $c \neq 0$. Then the above is a quadratic with complex coefficients, hence has at least one complex root (in fact, two counting multiplicity). Thus T has at least one finite fixed point. **Case 2:** $c = 0$. Then

$$T(z) = \frac{a}{d}z + \frac{b}{d} = \alpha z + \beta, \quad \alpha \neq 0.$$

The fixed point equation is

$$z = \alpha z + \beta \iff (\alpha - 1)z + \beta = 0.$$

If $\alpha \neq 1$, then

$$z = -\frac{\beta}{\alpha - 1}$$

is a finite fixed point. If $\alpha = 1$, then $T(z) = z + \beta$ with $\beta \neq 0$, which has no finite fixed point; however, on the Riemann sphere,

$$T(\infty) = \infty,$$

so ∞ is a fixed point. Therefore, every Möbius transformation has at least one fixed point on the Riemann sphere, and unless it is a nontrivial translation $z \mapsto z + \beta$, it has a finite fixed point.

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THE END