

Analytic functions and incompressible irrotational flow.

Let $\mathbf{v}$ be a vector field defined on a domain $D \subset \mathbb{R}^2$, we have that

$$\mathbf{v}(x, y) = (P(x, y), Q(x, y)).$$

The vector field describes a stationary flow where the “particles” move along paths (also called streamlines) that are tangent to $\mathbf{v}$. The incompressibility of the flow is described by the equation

$$\operatorname{div} \mathbf{v} = 0,$$

and the flow being irrotational is described by

$$\operatorname{curl} \mathbf{v} = 0.$$

Here we have

$$\operatorname{div} \mathbf{v} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \text{ and } \operatorname{curl} \mathbf{v} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}.$$

Define

$$f(z) = P(x, y) - iQ(x, y), \quad z = x + iy \in D.$$

Now the Cauchy-Riemann equations hold for f (**Check this!**) and thus f is analytic, specifically, f has an antiderivative, denote this by F . Now F is also analytic, and F is called *the complex velocity potential*, and we have that

$$\mathbf{v}(x, y) = \overline{F'(z)}, \quad z = x + iy.$$

We conclude that all analytic functions on a domain $D \subset \mathbb{C}$ will represent an incompressible and irrotational flow.

Problem 1. Draw the velocity field and sketch streamlines for: **a)** $F(z) = cz$, **b)** $F(z) = z^2/2$, **c)** $F(z) = 1/z$, **d)** $F(z) = \log(z)$, **e)** $F(z) = i \log(z)$.

We have that

$$F(z) = \varphi(x, y) + i\psi(x, y).$$

Since φ and ψ are harmonic conjugates, we have that $\nabla\varphi \cdot \nabla\psi = 0$ (**Check this!**). It follows that

$$\overline{F'(z)} = \overline{\frac{\partial\varphi}{\partial x} + i\frac{\partial\psi}{\partial x}} = \frac{\partial\varphi}{\partial x} - i\frac{\partial\psi}{\partial x} = \frac{\partial\varphi}{\partial x} + i\frac{\partial\varphi}{\partial y} = P + iQ.$$

Thus we get

$$\mathbf{v} = \nabla\varphi.$$

The streamlines are tangent to $\mathbf{v} = \nabla\varphi$, and $\nabla\psi$ is orthogonal to the level curves of ψ . The level curves of ψ are tangent to the streamlines and therefore the streamlines are described by the level curves of ψ ,

$$\text{streamlines} = \{(x, y) \in D \mid \psi(x, y) = c\}.$$

Problem 2. Find the streamlines for all the examples in the previous problem.

Assume $U \subset \mathbb{C}$ and F is analytic on U such that F defines a flow on U . If $g : D \rightarrow U$ is analytic, then $h = F \circ g$ will be analytic, and thus defines a flow on D .

Flow in a sector. This observation can be used for $U = \{\operatorname{Im} z > 0\}$, $F(z) = z$ and $D = \{z \mid 0 < \arg(z) < \alpha\}$ where $0 < \alpha < 2\pi$. Now we find a h that maps D to U and is analytic. It is not difficult to see that

$$h(z) = z^{\pi/\alpha}$$

does the job if we select a branch that is analytic in D (make a cut in the rest of the sector). We get that

$$h(z) = e^{\frac{\pi}{\alpha} \operatorname{Log}|z|} e^{i \frac{\pi}{\alpha} \arg(z)} = |z|^{\frac{\pi}{\alpha}} \cos\left(\frac{\pi}{\alpha} \arg(z)\right) + i |z|^{\frac{\pi}{\alpha}} \sin\left(\frac{\pi}{\alpha} \arg(z)\right).$$

Expressed in polar coordinates, the streamlines are therefore given by

$$r = c \left[\sin\left(\frac{\pi}{\alpha} \theta\right) \right]^{-\frac{\alpha}{\pi}}$$

Problem 3. Sketch the streamlines. What is $\lim_{D \ni z \rightarrow 0} |\mathbf{v}(z)|$?

Flow around a cylinder. First assume that the flow is symmetric about the x -axis. We can simplify somewhat by considering the flow around a “half” cylinder and defining D by

$$D = \{|z| > 1\} \cap \{\operatorname{Im}(z) > 0\}. \quad (\text{Draw } D!)$$

Let U be as before and set

$$h(z) = z + \frac{1}{z}.$$

Problem 4.

a) Show that $h(\text{the boundary of } D) = \text{the boundary of } U$.

b) Show that $h(D) = U$.

Thus h defines a flow in D . We have that

$$h(z) = x + iy + \frac{x - iy}{x^2 + y^2},$$

so the streamlines are given by

$$y \left(1 - \frac{1}{x^2 + y^2} \right) = c$$

in Cartesian coordinates and

$$r = \frac{c}{\sin(\theta)} \left(1 + \sqrt{1 + \frac{\sin^2(\theta)}{|c|}} \right)$$

in polar coordinates. These formulas also give the streamlines for $y < 0$, (then we solve with $c < 0$).

Problem 5. Draw some streamlines.

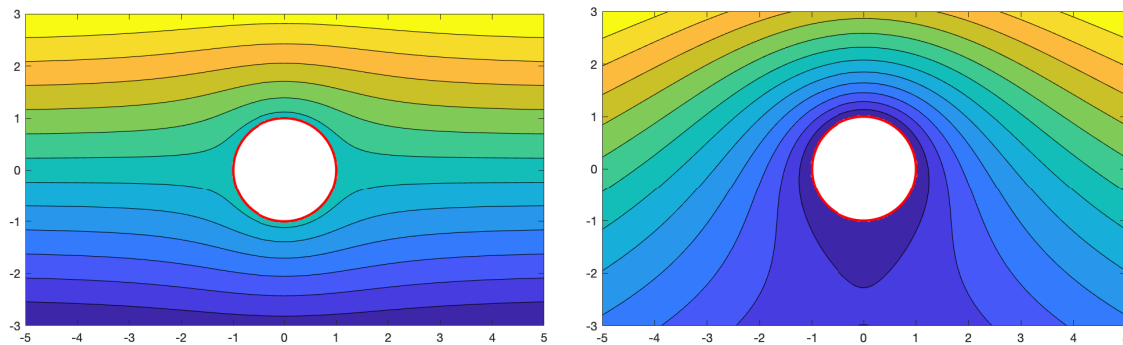
If we want to find a flow that goes around the cylinder, and which is not symmetric with respect to the real axis, we can consider

$$g(z) = h(z) + Ci \operatorname{Log}(z),$$

where C is a real constant. This gives a flow around the cylinder since both h and $i \operatorname{Log}(z)$ have $|z| = 1$ as a streamline. Now we have

$$\psi(x, y) = y \left(1 - \frac{1}{x^2 + y^2} \right) + \frac{C}{2} \operatorname{Log}(x^2 + y^2).$$

Below you see streamlines for symmetric flow ($c = 0$) and asymmetric flow ($c = 3$) around the cylinder.



Жюковский airfoils. Let $R > 0$ and $z_0 \in \mathbb{C}$ be fixed and define the function

$$J(z) = h(Rz + z_0),$$

where $h(z) = z + 1/z$. We have that J is analytic with a simple pole at $z = -z_0/R$. It turns out that for small z_0 , J maps the unit circle to an “airfoil-like” closed loop.



If we map the flow around the unit circle C_1 using J we get a flow around the airfoil. The figure below shows an example of this with $z_0 = -0.13 + 0.225i$ and $R = 1.15$. These so-called *Joukowski-airfoils* have been widely used to design airplane wings.

