3rd November, 2026

MAT2410 — Introduction to complex analysis

Mandatory assignment 2 of 2

Submission deadline

Thursday 29 October 2026, 14:30 in Canvas (canvas.uio.no).

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

You may answer in Norwegian or English. Every submitted solution will be awarded a passing grade. Collaboration is permitted and encouraged; LLM assistance is permitted but strongly discouraged.

Problem 1. Let $f(z) = \frac{e^z}{(1-z)^2}$, and let $\sum_{k=0}^{\infty} a_k z^k$ be the power series expansion of f around 0.

(a) Determine a_0, a_1 and a_2.

(b) Determine the radius of convergence of the power series.

(c) Compute

$$\int_{|z|=2} f(z)\,dz,$$

where the path around the circle $|z| = 2$ is oriented counterclockwise.

Solution. **(a)** We have power series expansions at 0

$$e^z = 1 + z + \frac{z^2}{2} + \dots$$

and

$$(1-z)^{-1} = 1 + z + z^2 + \dots,$$

which gives

$$(1-z)^{-2} = (1 + z + z^2 + \dots)^2 = 1 + 2z + 3z^2 + \dots,$$

where $\dots$ denotes terms of order 3 and higher. Multiplying these we get

$$\frac{e^z}{(1-z)^2} = (1 + z + \frac{z^2}{2} + \dots)(1 + 2z + 3z^2 + \dots) = 1 + 3z + \frac{11}{2}z^2 + \dots,$$

so $a_0 = 1$, $a_1 = 3$, and $a_2 = \frac{11}{2}$.

Alternative (harder) solution: Differentiating $f(z)$ we find

$$f'(z) = e^z \frac{3-z}{(1-z)^3}$$

and

$$f''(z) = e^z \frac{z^2 - 6z + 11}{(1-z)^4},$$

and so get

$$a_0 = f(0) = 1, \quad a_1 = f'(0) = 3, \quad a_2 = \frac{f''(0)}{2} = \frac{11}{2}.$$

(b) The function is analytic on the disc $\{z \in \mathbb{C} \mid |z| < 1\}$, and so the radius of convergence is at least 1. When $z \to 1$, $f(z) \to \infty$, so f cannot be extended to be analytic at 1. Thus the radius of convergence is exactly 1.

(c) The function e^z is entire (that is, analytic on all of $\mathbb{C}$). By Cauchy's integral formula, we have

$$\frac{d}{dz}\left[e^z\right]_{z=1} = \frac{1}{2\pi i}\oint_{|z|=2} \frac{e^z}{(z-1)^2}\,dz = \frac{1}{2\pi i}\oint_{|z|=2} f(z)\,dz.$$

Since $\frac{d}{dz}\left[e^z\right]_{z=1} = e^1 = e$, we find

$$\oint_{|z|=2} f(z)\,dz = 2\pi i e.$$

□

Problem 2. Let $D \subseteq \mathbb{C}$ be a bounded domain with piecewise smooth boundary. Let f be analytic on D, extending smoothly to $\overline{D}$.

(a) Show that if f is constant on ∂D, then f is constant on $\overline{D}$.

(b) Show that if $f(z) \neq 0$ for all $z \in \partial D$ and

$$\int_{\partial D} \frac{1}{f(z)}\,dz \neq 0,$$

then f has a zero in D.

Solution. **(a)** Let $c \in \mathbb{C}$ be such that $f(z) = c$ for all $z \in \partial D$, and let $g(z) = f(z) - c$ for all $z \in \overline{D}$. Then g is analytic on D, extending smoothly to $\overline{D}$, and moreover $g(z) = 0$ for all $z \in \partial D$. By the maximum principle, $|g|$ attains its maximal value on ∂D, hence $g(z) = 0$ for all $z \in \overline{D}$, and so $f(z) = c$ for all $z \in \overline{D}$.

Alternative solution: By the Cauchy integral formula, we have, for $w \in D$, that

$$f'(w) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(z)}{(z-w)^2}\,dz = \frac{1}{2\pi i}\int_{\partial D} \frac{c}{(z-w)^2}\,dz.$$

Since $\frac{c}{(z-w)^2}$ has a primitive $\frac{-c}{z-w}$ on $\mathbb{C} \setminus \{w\}$, the integral vanishes, so $f'(w) = 0$. Since this is true for all $w \in D$ and D is a domain, we have that f is constant on D, and so by continuity on $\overline{D}$.

(b) Suppose for a contradiction that f has no zeros in D, and therefore not in $\overline{D}$. Then $g(z) = \frac{1}{f(z)}$ is analytic on D, extending smoothly to $\overline{D}$, and so

$$\int_{\partial D} \frac{1}{f(z)}\,dz = \int_{\partial D} g(z)\,dz = 0,$$

by Cauchy's theorem. This gives a contradiction. □

Problem 3. Let $\alpha \in \mathbb{C}$. For each integer $k \geq 1$, let

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!},$$

and let

$$\binom{\alpha}{0} = 1.$$

Consider the power series

$$f(z) = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k.$$

(a) Determine the radius of convergence of the power series; the answer will depend on whether $\alpha \in \mathbb{Z}_{\geq 0}$ or not.

(b) Recall that for $w \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ we define the principal branch P_α of the α-th power function w^α by

$$P_\alpha(w) = e^{\alpha \operatorname{Log}(w)},$$

where $\operatorname{Log}(w)$ denotes the principal branch of the logarithm. Show that

$$P'_\alpha(w) = \alpha P_{\alpha-1}(w)$$

for all $w \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

(c) Determine $P_\alpha^{(k)}(1)$ for all $k \geq 0$ and use this to show that for $|z| < 1$, we have

$$f(z) = P_\alpha(1+z).$$

Solution. **(a)** Suppose first that $\alpha \in \mathbb{Z}_{\geq 0}$. Then $\binom{\alpha}{k} = 0$ for $k > \alpha$, since it contains a factor $(\alpha - \alpha) = 0$. Thus the power series is a polynomial, so converges everywhere, and the radius of convergence is ∞.

Suppose next that $\alpha \notin \mathbb{Z}_{\geq 0}$. Then $\binom{\alpha}{k} \neq 0$ for all $k \geq 0$, and we can apply the ratio test to the fraction

$$\frac{\binom{\alpha}{k}}{\binom{\alpha}{k+1}} = \frac{\frac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!}}{\frac{\alpha(\alpha-1)\dots(\alpha-k)}{(k+1)!}} = \frac{k+1}{\alpha-k} = -1 + \frac{1+\alpha}{\alpha-k}.$$

Since $\frac{1+\alpha}{\alpha-k} \to 0$ as $k \to \infty$, we have

$$\left|\frac{\binom{\alpha}{k}}{\binom{\alpha}{k+1}}\right| = \left|-1 + \frac{1+\alpha}{\alpha-k}\right| \to 1$$

as $k \to \infty$. By the ratio test, then, the radius of convergence is 1.

(b) We know that $\operatorname{Log}'(w) = \frac{1}{w}$ and that $e^{\operatorname{Log}(w)} = w$ for all $w \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Applying the chain rule, we get

$$\begin{aligned} P'_\alpha(w) &= \alpha \operatorname{Log}'(w) e^{\alpha \operatorname{Log}(w)} = \frac{\alpha}{w} e^{\alpha \operatorname{Log}(w)} = \frac{\alpha}{e^{\operatorname{Log}(w)}} e^{\alpha \operatorname{Log}(w)} \\ &= \alpha e^{\alpha \operatorname{Log}(w) - \operatorname{Log}(w)} = \alpha e^{(\alpha-1)\operatorname{Log}(w)} = \alpha P_{\alpha-1}(w). \end{aligned}$$

(c) We claim that $P_\alpha^{(0)}(1) = 1$, and for all $k \geq 1$, $P_\alpha^{(k)}(1) = \alpha(\alpha-1)\dots(\alpha-k+1)$. We prove this by induction on k, starting from the base case $k = 0$, which is true by

$$P_\alpha(1) = e^{\alpha \operatorname{Log}(1)} = e^{\alpha \cdot 0} = e^0 = 1.$$

Assume now the claim holds (for all α) for some $k \geq 0$; the induction step is to prove it holds for $k+1$. By problem (b), we have $P_\alpha^{(k+1)}(1) = \alpha P_{\alpha-1}^{(k)}(1)$. If $k = 0$, this gives $P_\alpha^{(1)}(1) = \alpha$ as claimed, while if $k > 0$, we find $P_\alpha^{(k+1)}(1) = \alpha(\alpha-1)\dots(\alpha-k)$, which again is what we wanted. Therefore the claim holds for all $k \geq 0$.

It follows that

$$\binom{\alpha}{k} = \frac{P_\alpha^{(k)}(1)}{k!},$$

and so the power series expansion of $P_\alpha(w)$ at $w = 1$ is

$$P_\alpha(w) = \sum_{k=0}^{\infty} \frac{P_\alpha^{(k)}(1)}{k!}(w-1)^k = \sum_{k=0}^{\infty} \binom{\alpha}{k}(w-1)^k,$$

where the power series converges to $P_\alpha(w)$ for all w with $|w-1| < 1$, since P_α is analytic on the set $\{w \mid |w-1| < 1\}$. Substituting $w = 1+z$ gives

$$P_\alpha(1+z) = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k = f(z)$$

whenever $|z| < 1$. □

Problem 4. Let $f(z) = \frac{e^z}{z}$, let $r > 0$, and let γ_r be the path

$$\gamma_r(t) = -r \sin t + ir \cos t, \qquad 0 \leq t \leq \pi.$$

Show that

$$\left| \int_{\gamma_r} f(z)\, dz \right| \leq \pi.$$

Solution. The path γ_r forms a half circle centred at 0 of radius r, starting at ir, through $-r$ and ending at $-ir$. Its length is $L = \pi r$. For every $z = \gamma_r(t)$ with $t \in [0, \pi]$, we have

$$|f(z)| = \left|\frac{e^z}{z}\right| = \frac{|e^z|}{|z|} = \frac{e^{\mathrm{Re}(z)}}{r} = \frac{e^{-r\sin(t)}}{r}.$$

As $\sin(t) \geq 0$, we have $e^{-r\sin(t)} \leq 1$, and hence

$$|f(z)| \leq \frac{1}{r}.$$

Taking $M = \frac{1}{r}$, the ML estimate now gives

$$\left|\int_{\gamma_r} f(z)\,dz\right| \leq ML = \frac{1}{r}\pi r = \pi.$$

Remark: For $r > 1$, the bound can be improved to

$$\left|\int_{\gamma_r} f(z)\,dz\right| \leq \frac{\pi}{r},$$

by Jordan's lemma [Gamelin, VII.7]. □

Problem 5. For each of the hypothetical function descriptions below, either give an example of a function satisfying the description, or prove that no such function can exist.

(a) A function which is defined and analytic on $\mathbb{C} \setminus \mathbb{Z}$, but does not admit an analytic extension to any point of $\mathbb{Z}$.

(b) An entire function such that $|f^{(k)}(0)| \geq k!$ for all $k \geq 0$.

(c) An entire function such that $f\left(\frac{1}{n}\right) = \frac{1}{2-1/n}$ for all integers $n \geq 1$.

Solution. **(a)** The function $f(z) = \frac{1}{\sin(\pi z)}$ is analytic wherever $\sin(\pi z) \neq 0$, which is whenever $z \notin \mathbb{Z}$. For each $z_0 \in \mathbb{Z}$, we have $\lim_{z\to z_0} \sin(\pi z) = 0$, and so $\lim_{z\to z_0} f(z) = \infty$. It follows that f cannot be extended analytically (or even continuously) over any point of $\mathbb{Z}$.

(b) No such function can exist. Assume for a contradiction that such a function f exists, and let

$$\sum_{k=0}^{\infty} a_k z^k$$

be its power series expansion at 0. We then have

$$|a_k| = \left|\frac{f^{(k)}(0)}{k!}\right| = \frac{\left|f^{(k)}(0)\right|}{k!} \geq \frac{k!}{k!} = 1$$

for all k. It follows that for any $r > 1$, the sequence $|a_k|r^k$ is unbounded, and therefore the convergence radius of the power series is at most 1. This is a contradiction, as an entire function has a power series with convergence radius ∞ at every point.

(c) No such function can exist. Assume for a contradiction that such a function f exists, and let

$$g(z) = f(z) - \frac{1}{2-z},$$

for all $z \neq 2$. Then g is analytic on the domain $\mathbb{C} \setminus \{2\}$, and has a zero at $1/n$ for every $n \geq 1$. By continuity $g(0) = 0$, and so g has a non-isolated zero at 0, from which it follows that $g(z) = 0$ for all $z \neq 2$. This means that $f(z) = \frac{1}{2-z}$ for all $z \in \mathbb{C} \setminus \{2\}$. We then have $\lim_{z\to 2} f(z) = \infty$, which contradicts f being entire. □