

MAT2410-Introduction to complex analysis, Sec. I.1

A complex number is formally given by a pair (x, y) of real numbers, usually written $x+iy$.

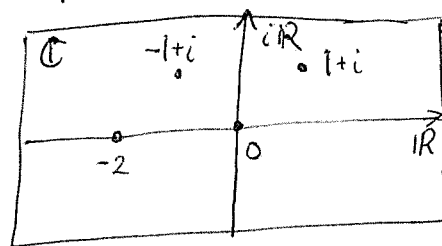
For $z = x+iy$, have

- x is the real part of z , denoted $x = \operatorname{Re} z$
- y is the imaginary part of z , denoted $y = \operatorname{Im} z$.

Represent a complex no. $z = x+iy$ by the point (x, y) in the plane, call the set of all cpx nos

$$\mathbb{C} = \{x+iy \mid x, y \in \mathbb{R}\}$$

the complex plane.



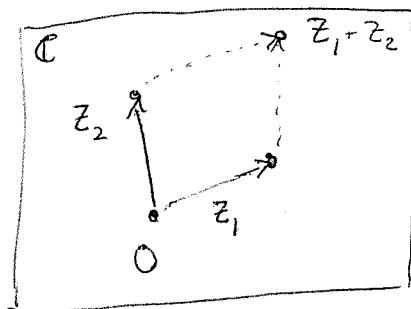
Have subsets

$$\mathbb{R} = \{x+0i \mid x \in \mathbb{R}\} \subseteq \mathbb{C}, \text{ the } \underline{\text{real axis}}$$

$$i\mathbb{R} = \{0+iy \mid y \in \mathbb{R}\} \subseteq \mathbb{C}, \text{ the } \underline{\text{imaginary axis}}$$

Addition of cpx nos as for vectors:

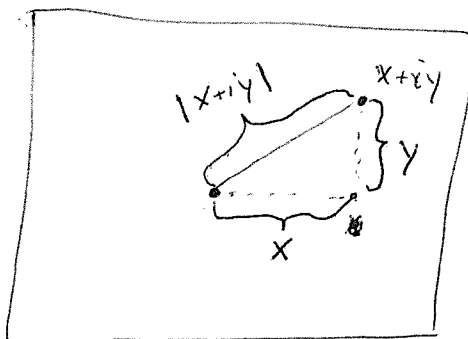
$$(x_1+iy_1) + (x_2+iy_2) = (x_1+x_2) + i(y_1+y_2)$$



Modulus of a complex no. z is the distance between z & 0 in the plane, written $|z|$

$$z = x + iy$$

$$|z| = \sqrt{x^2 + y^2}$$

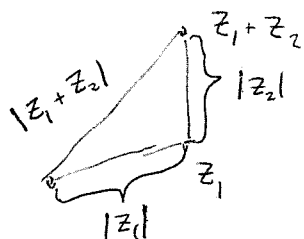


We will often need to estimate $|z|$!

Primary tool: Triangle inequality:

Thm 1 If $z_1, z_2 \in \mathbb{C}$, then $|z_1 + z_2| \leq |z_1| + |z_2|$ lengths

Pf:



$|z_1|, |z_2|, |z_1 + z_2|$ are sides in a triangle

Question: When is $|z_1 + z_2| = |z_1| + |z_2|$?

Thm 2 If $z_1, z_2 \in \mathbb{C}$, then $|z_1 - z_2| \geq |z_1| - |z_2|$.

Pf Equivalent to $|z_1 - z_2| + |z_2| \geq |z_1|$, true by thm 1.

Multiplication of complex nos defined by

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

Informally, "compute as usual & use $i^2 = -1$ ".

Then, for $z_1, z_2, z_3 \in \mathbb{C}$

- $1z_1 = z_1$
- $z_1z_2 = z_2z_1$
- $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$,

i.e. usual rules of algebra apply ($\mathbb{C}$ is a "ring").

Have division, if $z = x + iy$, $w = u + iv$, $w \neq 0$, then the inverse of w is

$$w^{-1} = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u-iv} \cdot \frac{1}{u+iv} = \frac{u}{u^2+v^2} + i \frac{-v}{u^2+v^2},$$

$$\& \quad \frac{z}{w} = z \cdot \frac{1}{w} = \frac{ux+vy}{u^2+v^2} + i \frac{uy-vx}{u^2+v^2},$$

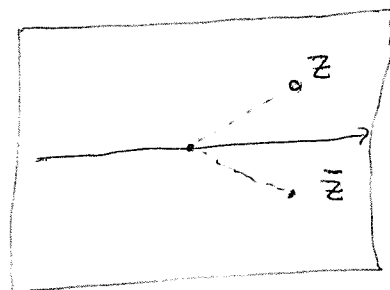
$$w \cdot \frac{1}{w} = 1$$

$$w \cdot \frac{z}{w} = z.$$

This means $\mathbb{C}$ is a "field".

~~The~~ The complex conjugate of ~~z~~ $z = x + iy \in \mathbb{C}$ is

$$\bar{z} = x - iy$$



~~Useful~~ Useful relations:

$$\bullet \quad z \bar{z} = |z|^2$$

$$\bullet \quad \frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

$$\bullet \quad |\bar{z}| = |z|$$

$$\bullet \quad \overline{z+w} = \bar{z} + \bar{w}$$

$$\bullet \quad \overline{z \cdot w} = \bar{z} \bar{w}$$

$$\bullet \quad |z|^2 |w|^2 = z \bar{z} w \bar{w} = (zw)(\bar{z}\bar{w}) = (zw)(\overline{zw}) = |zw|^2 \Rightarrow |z||w| = |zw|.$$

What's in this course: Study complex-valued functions $f(z)$, where $z \in \mathbb{C}$ $f(z) \in \mathbb{C}$, which are differentiable in a complex sense

(analytic): $f'(z) = \lim_{h \rightarrow 0} \frac{f(z+h) - f(z)}{h}$ exists for $h \in \mathbb{C}$.

Examples: z^n , $\sin(z)$, $\frac{1}{1-z}$, e^z , ...

These have surprising properties:

- Complex differentiable $\Rightarrow$ infinitely complex differentiable
- $f(z)$ analytic $\Rightarrow f: \mathbb{C} \rightarrow \mathbb{C}$ preserves angles
- $f(z)$ analytic $\Rightarrow f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ near any $z_0 \in \mathbb{C}$
- $f(z)$ analytic $\Rightarrow |f(z)|$ never obtains ~~the~~ maximum
- $f(z)$ analytic $\Rightarrow$ value of $f(z)$ at any point z_0 fully determined by knowing $f(z)$ near another point z_1
- $f(z)$ analytic $\Rightarrow f(z_0)$ computable from knowing $f(z)$ in a circle around z_0



Applications to Real Analysis, e.g. computation

We'll study integrals of analytic $f(z)$, $\int f(z) dz$,
& obtain surprising formulas for these



Applications: • Real analysis, e.g. definite integrals

$$\int_0^{2\pi} \frac{\cos \theta}{2 + \cos \theta} d\theta = 2\pi \left(1 - \frac{2}{3}\right), \quad \int_{-\infty}^{\infty} \frac{\cos(x)}{1+x^2} dx = \frac{\pi}{e}$$

- Number theory (not in this course), Riemann zeta fun., prime no. thm.
- Physics (fluid mechanics in this course, but many other places)
- But probably also most parts of mathematics...

Polynomials A complex polynomial is a function of the form

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 \quad z, a_i \in \mathbb{C}.$$

The fundamental theorem of algebra says if $n = \deg f(z) \geq 1$,
can factor

$$f(z) = a_n \prod_{i=1}^n (z - r_i) \quad r_i \in \mathbb{C},$$

Here $r_1, \dots, r_n$ are roots of $f(z)$, & $f(z)=0 \iff z \in \{r_1, \dots, r_n\}$.

Partial proof: By induction on n , from $n=1$: $f(z)=a_1z+a_0=a_1(z+\frac{a_0}{a_1})$

~~Assume~~ For $\deg f(z)=n>1$, assume $\exists r \in \mathbb{C}$ s.t. $f(r)=0$. $\checkmark$
Then can show (exercise!) there exists a polynomial $g(z)$ s.t.

$$f(z)=(z-r)g(z)$$

$$\deg g(z)=n-1 \xRightarrow{\text{induction}} g(z)=\frac{a_n}{a_n} \prod_{i=1}^{n-1} (z-r_i) \Rightarrow f(z)=a_n(z-r) \prod_{i=1}^{n-1} (z-r_i) \quad \square$$

Hard part is proving assumption $\circledast$: All polynomials $f(z)$, $\deg f(z) \geq 1$ have a complex root. We will do this $\S$ several ways in the course...

Note FTA fails for real polynomials, $f(x)=x^2+1$ has no real root and does not factor, $f(z)=z^2+1=(z+i)(z-i)$.