

I.2 & I.3

$x, y \in \mathbb{R}$

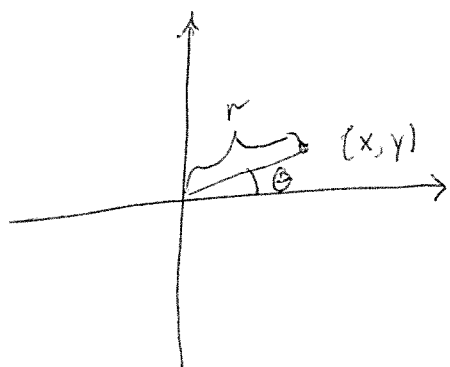
Recall a point $(x, y) \neq (0, 0)$ in the plane can be given in polar coordinates (r, θ) , with

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$r = \sqrt{x^2 + y^2}$$

θ = angle between ~~axis~~
positive x-axis &
vector (x, y)



This gives polar representation of complex numbers:

$$(*) \quad z = x + iy = r(\cos \theta + i \sin \theta),$$

with $r = |z|$ & θ called the

argument of z , written $\arg z = \theta$.

Note: • Only give polar rep. of z if $z \neq 0$

• $(*)$ only partially determines θ , since for any $k \in \mathbb{Z}$

$$\cos \theta = \cos(\theta + k \cdot 2\pi), \quad \sin \theta = \sin(\theta + k \cdot 2\pi),$$

i.e. any choice of $\theta = \arg z$ could be replaced with $\theta + k \cdot 2\pi$.

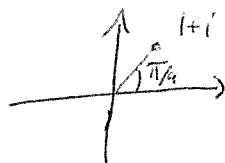
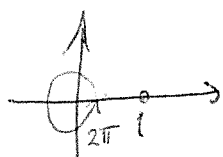
Def ~~A~~ Given sets X, Y , a multivalued function $f: X \rightarrow Y$ is a map which to each $x \in X$ assigns a non-empty subset $f(x) \subseteq Y$.

Note: Be suspicious of multivalued functions, don't assume results/concepts applicable to ordinary functions still apply.

Define $\arg: \mathbb{C} \setminus \{0\} \rightarrow \mathbb{R}$ as multivalued function

$$\arg(z) = \{ \theta \in \mathbb{R} \mid z = |z|(\cos \theta + i \sin \theta) \}$$

Ex $\arg(1) = \{ 2\pi k \mid k \in \mathbb{Z} \}$ $\arg(1+i) = \{ \frac{\pi}{4} + 2\pi k \mid k \in \mathbb{Z} \}$



Def ~~Let~~ Let $z \in \mathbb{C} \setminus \{0\}$. The principal value of $\arg(z)$, denoted $\text{Arg}(z)$, is the unique element $\theta \in \arg(z)$ with $-\pi < \theta \leq \pi$.

Then $\arg(z) = \{ \text{Arg}(z) + 2\pi k \mid k \in \mathbb{Z} \} = \text{Arg}(z) + 2\pi \mathbb{Z}$

Ex $\text{Arg}(1) = 0$, $\text{Arg}(1+i) = \frac{\pi}{4}$

Question: Is $\text{Arg}(z)$ continuous as a function $\mathbb{C} \setminus \{0\} \rightarrow \mathbb{R}$?

Def For any $\theta \in \mathbb{R}$, let

$$e^{i\theta} = \cos \theta + i \sin \theta$$

If $z \neq 0$, then with $r = |z|$, $\theta \in \arg(z)$, have

$$z = r(\cos \theta + i \sin \theta) = r e^{i\theta}$$

Addition formulas for $\cos$ & $\sin$

$$\cos(\theta + \varphi) = \cos \theta \cos \varphi - \sin \theta \sin \varphi$$

$$\sin(\theta + \varphi) = \cos \theta \sin \varphi + \sin \theta \cos \varphi$$

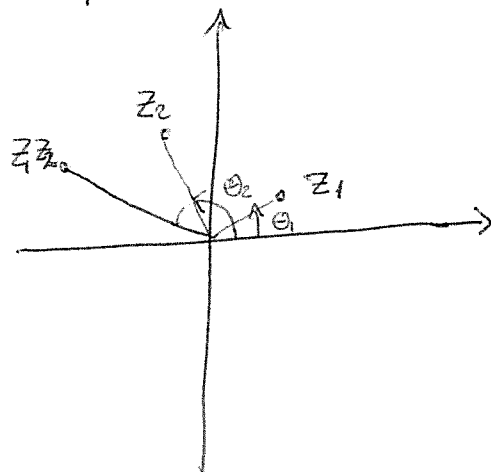
then give

$$e^{i(\theta + \varphi)} = e^{i\theta} e^{i\varphi}$$

This gives a geometric interpretation of multiplication

$$z_1 = r_1 e^{i\theta_1}$$

$$z_2 = r_2 e^{i\theta_2} \Rightarrow z_1 z_2 = (r_1 r_2) e^{i(\theta_1 + \theta_2)}$$



Lots of formulas: If $z = r e^{i\theta}$, then

$$\frac{1}{z} = \frac{1}{r} e^{-i\theta}$$

$$|e^{i\theta}| = 1$$

$$\bar{z} = r e^{-i\theta}$$

~~repeated~~

$$z^n = r^n e^{in\theta}$$

n-th roots: Let $n \geq 1$ be an integer, $w \in \mathbb{C}$, want to solve equation $z^n = w$ $z \in \mathbb{C}$.

Cases: I $w = 0$, $z^n = 0 \Leftrightarrow z = 0$.

II $w \neq 0$. Then $w = r e^{i\theta}$. Writing $z = s e^{i\varphi}$, have

$$z^n = w$$

$$\Updownarrow s^n e^{in\varphi} = r e^{i\theta}$$

$$\Updownarrow s^n = r \text{ and } e^{in\varphi} = e^{i\theta}$$

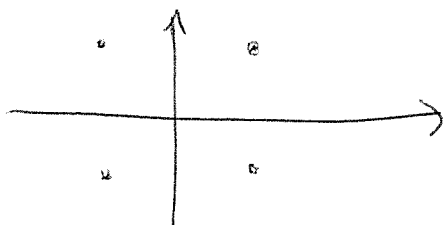
$$\Updownarrow s = r^{1/n}$$

$$n\varphi = \theta + 2\pi k \text{ for some } k \in \mathbb{Z}$$

$\leadsto$ n solutions, $z = r^{1/n} e^{i\frac{\theta + 2\pi k}{n}}$ $k = 0, 1, 2, \dots, n-1$

Ex $z^4 = 1$, $1 = e^{i0}$ solutions are

$$z = e^{i\frac{\pi}{4}}, e^{i\frac{3\pi}{4}}, e^{i\frac{5\pi}{4}}, e^{i\frac{7\pi}{4}} \quad \pm \frac{\sqrt{2}}{2} \pm i \frac{\sqrt{2}}{2}$$



Have a multivalued n -th root function

$$f(w) = \{ z \in \mathbb{C} \mid z^n = w \}$$

The extended complex plane is the set

$$\mathbb{C}^* = \mathbb{C} \cup \{\infty\},$$

where ∞ is called the point at infinity.

Where is ∞ ?

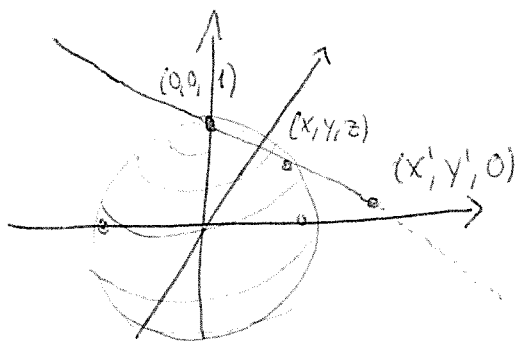
Visualise via the stereographic projection: Let S denote the unit sphere

in $\mathbb{R}^3$: $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, & define a map

$$S \rightarrow \mathbb{C}^* \text{ by}$$

$$(0, 0, 1) \mapsto \infty$$

$(x, y, z), z \neq 1 \mapsto$ the intersection of the line thru $(0, 0, 1)$ & (x, y, z) with the (x, y) -plane.



Conversely, ^{for} any $(x', y', 0)$, the line from $(x', y', 0)$ thru $(0, 0, 1)$ intersects S in 1 other point, so have a bijection

$$S \longleftrightarrow \mathbb{C}^*$$

Equations for the maps $S \longleftrightarrow \mathbb{C}^*$

$S \rightarrow \mathbb{C}^*$ Given $p = (x, y, z) \in S$, line through $(1, 0, 0)$ & p is

$$l = \{ t(0, 0, 1) + (1-t)(x, y, z) \mid t \in \mathbb{R} \}$$

If $(x', y', 0) \in l$, then have $t + (1-t)z = 0 \Rightarrow t = \frac{-z}{1-z}$

$$\Rightarrow 1-t = \frac{1}{1-z} \Rightarrow (x', y', 0) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0 \right).$$

Map $S \rightarrow \mathbb{C}^*$ given by $(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}, 0 \right) = \frac{x}{1-z} + i \frac{y}{1-z}$

$\mathbb{C}^* \rightarrow S$. Let $w = u + iv \in \mathbb{C}^*$, which $(x, y, z) \in S$ maps to w ?

Have $u = \frac{x}{1-z}, v = \frac{y}{1-z}, x^2 + y^2 + z^2 = 1.$

$$\Downarrow$$

$$x = u(1-z) \quad y = v(1-z), \quad u^2(1-z)^2 + v^2(1-z)^2 + z^2 = 1$$

Solve for $z \rightsquigarrow \underline{z = 1}$
not valid

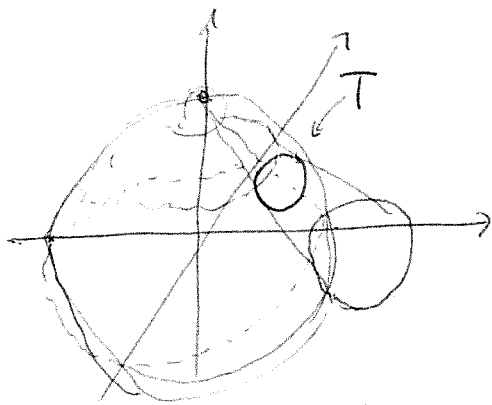
$$z = \frac{u^2 + v^2 - 1}{u^2 + v^2 + 1} = \frac{|w|^2 - 1}{|w|^2 + 1},$$

then obtain

$$(x, y, z) = \left(\frac{2u}{|w|^2 + 1}, \frac{2v}{|w|^2 + 1}, \frac{|w|^2 - 1}{|w|^2 + 1} \right)$$

as formula for map $\mathbb{C}^* \rightarrow S$.

Thm Under the correspondence $S \leftrightarrow \mathbb{C}^*$, ~~every map gives~~
circles in $S \leftrightarrow$ circles/lines in $\mathbb{C}^*$, where a circle
in S containing $(0,0,1)$ maps to a line, & a circle not containing
 $(0,0,1)$ maps to a circle.



Pf A circle $\cap$ in S is given by intersecting
 S with a plane, given say by eqn.

$$\alpha x + \beta y + \gamma z = \delta$$

Then $w = (u, v) \in \mathbb{C}^*$ lies in the image of T

if and only if $\alpha \frac{2u}{|w|^2 + 1} + \beta \frac{2v}{|w|^2 + 1} + \gamma \frac{|w|^2 - 1}{|w|^2 + 1} = \delta$ ($|w|^2 = u^2 + v^2$)

$$\Updownarrow \cdot (1 + |w|^2)$$

$$(\gamma - \delta)|w|^2 + 2\alpha u + 2\beta v - \delta - \gamma = 0$$

$$\Updownarrow$$

$(0,0,1) \in T \iff$ If $\gamma = \delta$: $2\alpha u + 2\beta v = \delta + \gamma$, a line

$(0,0,1) \notin T \iff$ If $\gamma \neq \delta$: ~~$(u + \frac{\alpha}{\gamma - \delta})^2 + (v + \frac{\beta}{\gamma - \delta})^2 = \frac{\gamma^2 + \alpha^2 + \beta^2 - \delta^2}{(\gamma - \delta)^2}$~~

a circle