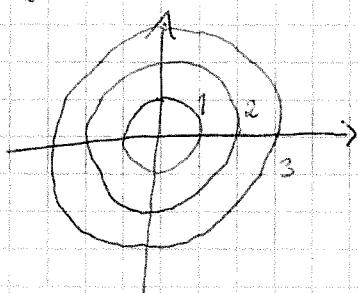


Visualising complex functions

Given a complex function  $f(z)$ , how to visualise its behaviour? Different options, recall given a real fun  $g(x, y)$ , can draw level curves:

$$g(x, y) = \sqrt{x^2 + y^2}$$



$$g(x, y) = xy$$

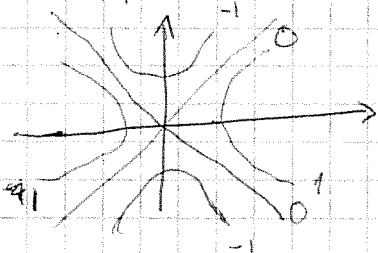


Option 1a)

A complex function  $f(z) = u(x, y) + i v(x, y)$  can be graphed by drawing level curves for  $u$  &  $v$  ( $u = \text{Re} f$ ,  $v = \text{Im} f$ )

Ex  $f(z) = z^2$ ,  $z = x + iy$   
 $= (x + iy)^2 = x^2 - y^2 + i 2xy$

$$u(x, y) = x^2 - y^2$$



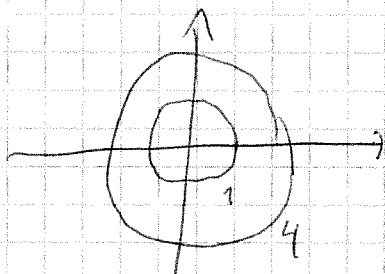
$$v(x, y) = 2xy$$

see above

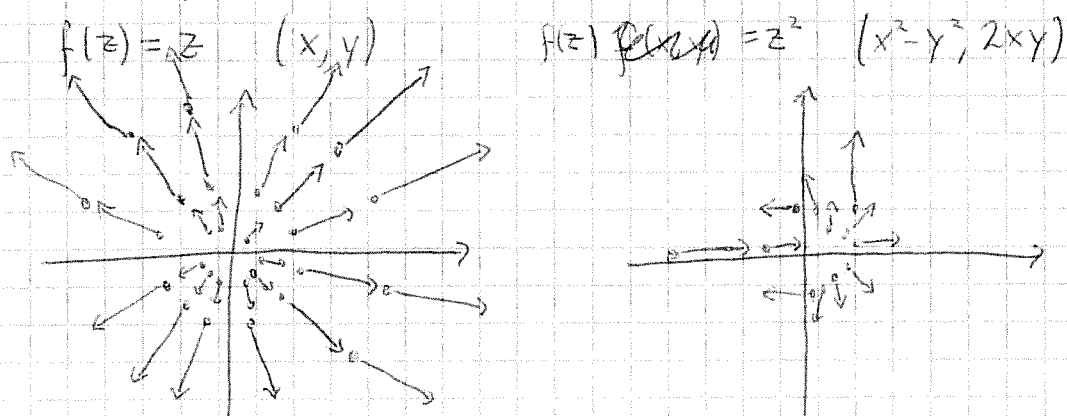
1b)

Often more useful: Level curves of  $|f(z)|$

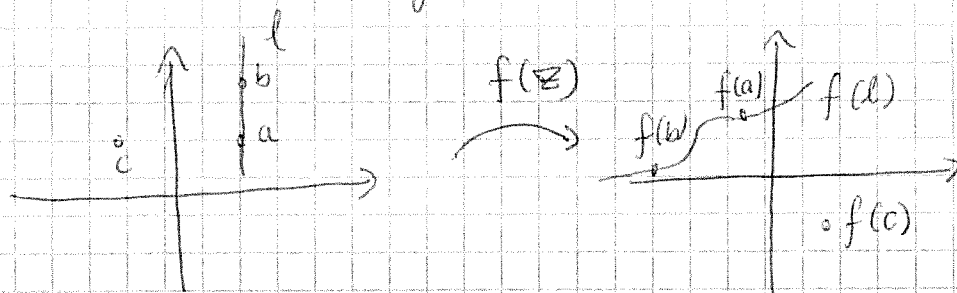
Ex  $f(z) = z^2$ ,  $|f(z)| = |z|^2 = x^2 + y^2$



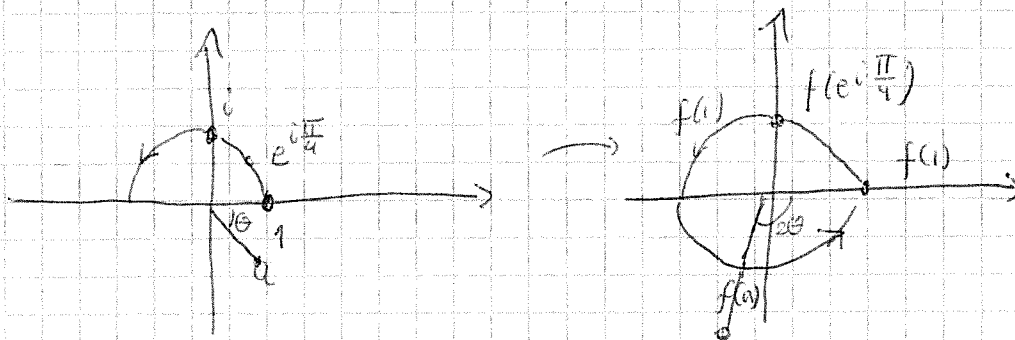
Option 2: Think of  $f(z) = u(x, y) + i v(x, y)$  as vector field  $(u(x, y), v(x, y))$ , draw these



Option 3: Draw 2 copies of  $\mathbb{C}$ , mark points, lines, circles in one, & their images in the other.

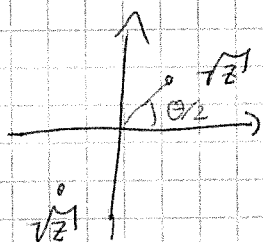
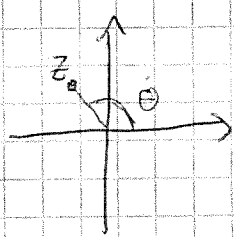


Ex  $f(z) = z^2$   $z = r e^{i\theta} \Rightarrow f(z) = z^2 = r^2 e^{i2\theta}$   
 $|f(z)| = |z|^2$   $\arg f(z) = 2 \arg z$



Square-root function Recall for  $w \in \mathbb{C}$ ,  $w \neq 0$ ,

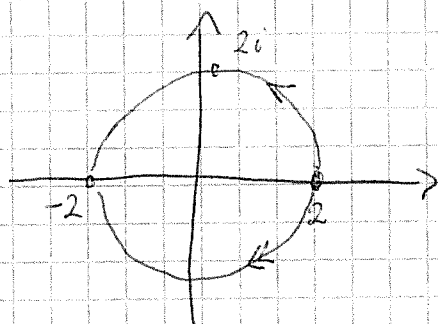
have 2 square roots,  $w = r e^{i\theta}$  sq. roots  
 $r^{1/2} e^{i\frac{\theta}{2}}$  &  $r^{1/2} e^{i\frac{\theta}{2} + \pi} = -r^{1/2} e^{i\frac{\theta}{2}}$ .



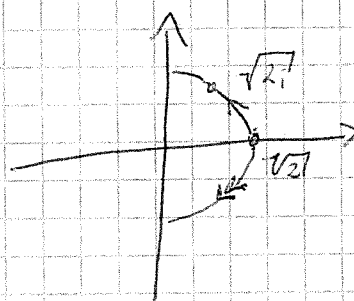
Want single-valued square root function, need rule for choosing between 2 roots.

Possible rule: Choose the square root with positive real part (i.e. in right half plane).

If  $z \notin \mathbb{R}_{<0}$ , i.e.  $\text{Arg } z \neq \pi$ , then sq. roots are not purely imaginary, so exactly one has pos. real part.



$\sqrt{z}$



No way to ~~define~~ extend this to continuous function on  $\mathbb{C}$ , only on  $\mathbb{C} \setminus \mathbb{R}_{<0}$ .

In formulas  $\sqrt{z} = |z|^{1/2} e^{i \frac{\text{Arg } z}{2}}$ .

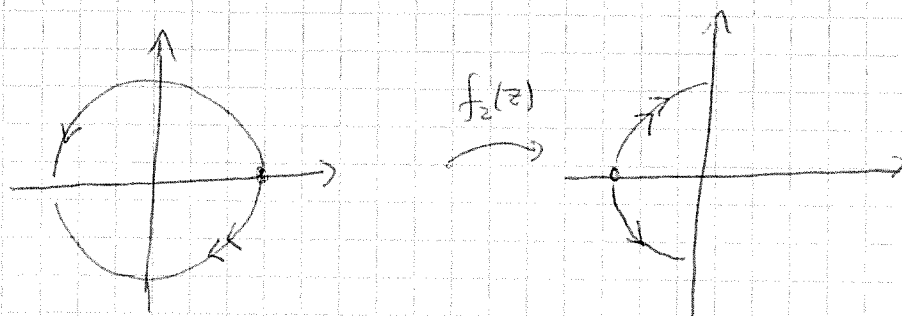
Def If  $f: A \rightarrow B$  is a multivalued function &  $g: A \rightarrow B$  is a function so that  $g(a) \in f(a)$  for all  $a$ , say  $g$  is a branch of  $f$ .

Note: Often require  $g$  to be continuous, terminology slightly informal.

Have removed  $\mathbb{R}_{<0}$  from  $\mathbb{C}$  to get a branch of square root function, this is a

Branch cut: Removing parts of the domain of multivalued function to get a (continuous) branch.

On  $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ , call  $f_1(z) = |z|^{1/2} e^{i \frac{\text{Arg}(z)}{2}}$  the principal branch, also have  $f_2(z) = -f_1(z)$ , a second branch.



Later study square root function via branch cuts & branches.

Alternative approach (not used by us): Define  $\sqrt{z}$  on a new space; 2 copies of  $\mathbb{C} \setminus \{0\}$  glued so that passing  $\mathbb{R}_{\leq 0}$  switches which copy you are on. This gives a "Riemann surface"  $S$ , & well-defined function  $\sqrt{z}: S \rightarrow \mathbb{C} \setminus \{0\}$ .

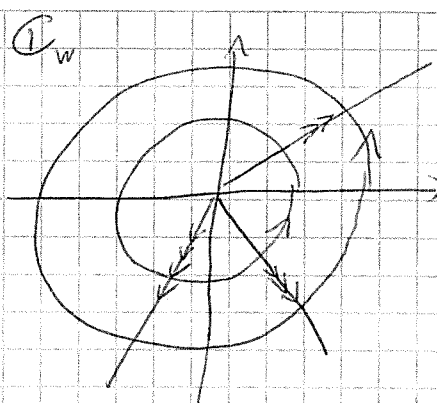
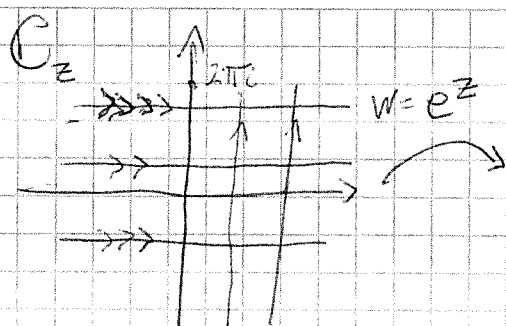
Exponential function Have defined  $e^x e^{iy} = \cos y + i \sin y$  for  $x, y \in \mathbb{R}$ , now define for  $z = x + iy$

$$e^z = e^{x+iy} = e^x e^{iy}$$

Then, for  $z = x + iy$   $w = u + iv$ , have

$$\begin{aligned} e^{z+w} &= e^{x+u+i(y+v)} = e^{x+u} e^{i(y+v)} = e^x e^u e^{iy} e^{iv} \\ &= e^x e^{iy} e^u e^{iv} = e^z e^w. \end{aligned}$$

Since  $e^{iy} = e^{i(y+2\pi)}$ , have  $e^z = e^{z+2\pi i}$ , function  $z \mapsto e^z$  is  $2\pi i$ -periodic.



The logarithm Recall for  $x \in \mathbb{R}_{>0}$   $\log x \in \mathbb{R}$  is the unique number s.t.  $e^{\log x} = x$ .

For a complex number  $w \in \mathbb{C}$ , the equation

$$(*) \quad e^z = w$$

has  $\begin{cases} \text{no solution if } w=0 \\ \text{infinitely many solutions if } w \neq 0 \end{cases}$

$$z = x + iy$$

$$w = |w| e^{i \arg(w)} = e^z = e^x e^{iy} \Leftrightarrow e^x = |w| \text{ \& } e^{iy} = e^{i \arg(w)}$$

$$\Leftrightarrow x = \log |w| \text{ \& } y = \arg(w) + 2\pi k \quad k \in \mathbb{Z}$$

Solutions to

$$(*) \quad \text{Solutions are } z = \log |w| + i \arg(w) + 2\pi i k \quad k \in \mathbb{Z}.$$

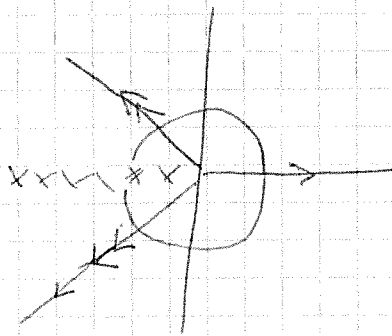
Def The multivalued logarithm function  $\log : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$  is

$$\log w = \{ z \in \mathbb{C} \mid e^z = w \} = \{ \log |w| + i \arg(w) + 2\pi i k \mid k \in \mathbb{Z} \}$$

The principal value of  $\log w$  is

$$\text{Log } w = \log |w| + i \arg(w).$$

On  $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ , this defines a branch of  $\log$



$$w = \text{Log } z$$

