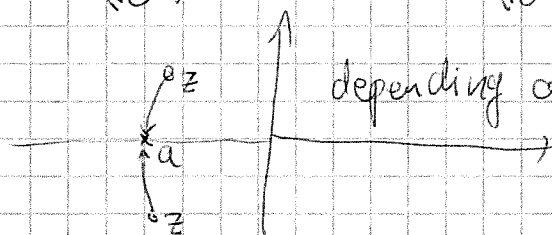


Function $\text{Log} : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$ is continuous except on $\mathbb{R}_{\leq 0}$, for $a \in \mathbb{R}_{>0}$ $z \rightarrow a$ gives $\text{Log}(z) \rightarrow \log_{\mathbb{R}}|a| \pm i\pi$, depending on $\text{Im } z \gtrless 0$.

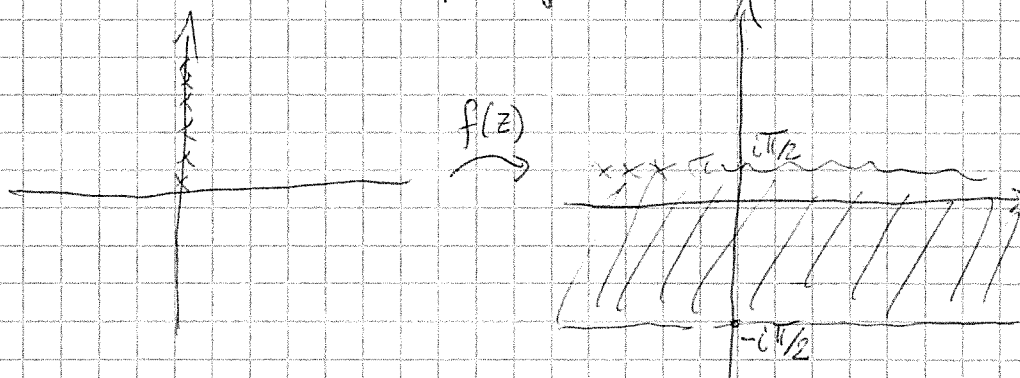


Make branch cut $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then get $\text{Log} : \mathbb{C} \setminus \mathbb{R}_{\leq 0} \rightarrow \mathbb{C}$, the principal branch of the multivalued log function.

Many branch cuts/branches possible, e.g.:

Branch cut $\mathbb{C} \setminus i\mathbb{R}_{\geq 0}$, branch

$f(z) = \log|z| + i\text{Arg}^*(z)$, where $\text{Arg}^*(z)$ is the unique value of $\arg(z)$ in $(-\frac{3\pi}{2}, \frac{\pi}{2})$.



~~Caution: If f is a branch of log, it is not necessarily true that $f(z_1 z_2) = f(z_1) + f(z_2)$, e.g.~~

Caution: If f is a branch of $\log$, it is not necessarily true that $f(z_1 z_2) = f(z_1) + f(z_2)$, e.g.

$$\text{Log}(e^{i\frac{3\pi}{4}} \cdot e^{i\frac{3\pi}{4}}) = \text{Log}(e^{i\frac{6\pi}{4}}) = i\text{Arg}(e^{i\frac{6\pi}{4}}) = -\frac{\pi}{2}$$

$$\text{Log}(e^{i\frac{3\pi}{4}}) + \text{Log}(e^{i\frac{3\pi}{4}}) = i\frac{3\pi}{4} + i\frac{3\pi}{4} = i\frac{3\pi}{2} \neq -\frac{\pi}{2}$$

(but $f(z_1 z_2) - f(z_1) - f(z_2) \in 2\pi i\mathbb{Z}$ (exercise)).

Recall for a complex number $z = x+iy$, we have

$$e^z = e^x e^{iy} = e^x (\cos y + i \sin y),$$

and so $e^z = e^{z+2\pi i k}$ for all $k \in \mathbb{Z}$.

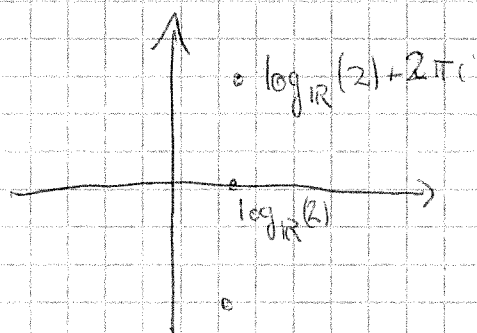
Have the multivalued log function, for $w \in \mathbb{C} \setminus \{0\}$

$$\begin{aligned} \log(u+iv) = \log w &= \{z \in \mathbb{C} \mid e^z = w\} \\ &= \{\log |w| + i \operatorname{Arg}(w) + 2\pi i k \mid k \in \mathbb{Z}\}. \end{aligned}$$

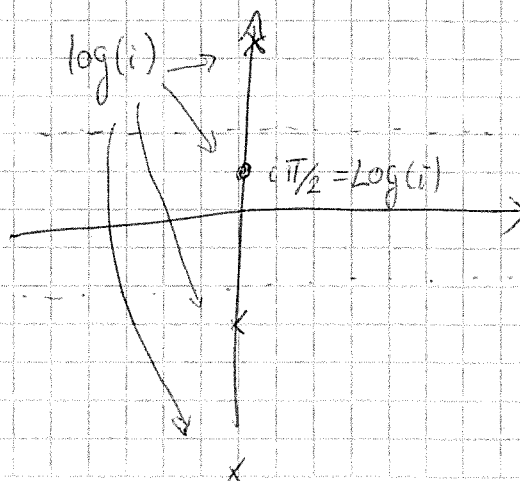
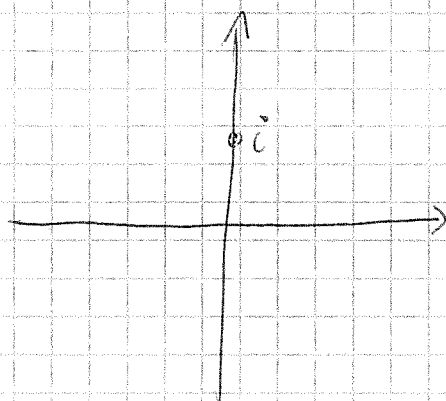
The principal value of the logarithm is

$$\operatorname{Log} w = \log |w| + i \operatorname{Arg}(w).$$

Ex $\log(2) = \{\log_{\mathbb{R}} 2 + i \operatorname{Arg}(2) + 2\pi i k \mid k \in \mathbb{Z}\}$
 $= \{\log_{\mathbb{R}} 2 + 2\pi i k \mid k \in \mathbb{Z}\}$



$$\begin{aligned} \operatorname{Log}(2) &= \log_{\mathbb{R}}(2) + i \operatorname{Arg}(2) \\ &= \log_{\mathbb{R}}(2). \end{aligned}$$



Power functions Recall if $x > 0$, α are real, then

$$x^\alpha = e^{\alpha \log_{\mathbb{R}}(x)}$$

This gives a definition of ~~exp~~ z^α for $z \in \mathbb{C} \setminus \{0\}$, $\alpha \in \mathbb{C}$,
i.e. have

$$z^\alpha = e^{\alpha \log(z)} = \{ e^{\alpha w} \mid w \in \log(z) \}$$

Note $\log(z)$ has multiple values,

$$\log(z) = \text{Log}(z) + 2\pi i k, \quad k \in \mathbb{Z}$$

so z^α does as well

$$z^\alpha = e^{\alpha(\text{Log}(z) + 2\pi i k)} = e^{\alpha \text{Log}(z)} e^{2\pi i \alpha k} = e^{\alpha \text{Log}(z)} (e^{2\pi i \alpha})^k, \quad k \in \mathbb{Z}.$$

Ex $i^i = ?$ $\text{Log}(i) = \log_{\mathbb{R}}|i| + i \text{Arg}(i) = \cancel{0} \log 1 + i \frac{\pi}{2}$

$$\alpha = i \leadsto e^{2\pi i \alpha} = e^{2\pi i^2} = e^{-2\pi}$$

$$i^i = e^{i \text{Log}(i)} (e^{-2\pi})^k = e^{-\frac{\pi}{2}} (e^{-2\pi})^k \quad k \in \mathbb{Z}$$

Choosing a branch $\overline{f(z)}$ of $\log$ on some branch cut $U \subseteq \mathbb{C}$
gives a branch $g(z)$ of the multivalued function $z \mapsto z^\alpha$ by

$$g(z) = e^{\alpha f(z)}$$

Ex On $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, have a branch $e^{\alpha \text{Log}(z)}$ of z^α -function.

Special cases: $\alpha \in \mathbb{Z}$. Then for $z \in \mathbb{C} \setminus \{0\}$, $k \in \mathbb{Z}$, have

$$e^{\alpha \text{Log}(z)} = e^{\alpha(\text{Log}(z) + 2\pi i k)} = e^{\alpha \text{Log}(z)} e^{2\pi i k \alpha} =$$

so z^α independent of value of $\log(z)$ chosen, z^α single-valued.

• $\alpha \in \mathbb{Q}$, $\alpha = \frac{p}{q}$. Then $(e^{2\pi i \frac{p}{q}})^k$ takes only q different values:

p, q coprime

if $w = e^{2\pi i \frac{p}{q}}$, then $w^q = e^{2\pi i (\frac{p}{q})q} = e^{2\pi i p} = 1$, so sequence w^k is

$$w^{q-1}, 1, w, w^2, \dots, w^{q-1}, w^q = 1, w, \dots$$

Hence z^α takes q values: $e^{k \log(z)} w^k$ $k=0, 1, \dots, q-1$.

• $z=e$, minor conflict of notation, ~~can~~ e^α can mean

$$e^\alpha = e^{\operatorname{Re} \alpha} (\cos(\operatorname{Im} \alpha) + i \sin(\operatorname{Im} \alpha)) \quad (\text{usually})$$

or ~~multivalued~~ or multivalued function z^α evaluated at $z=e$.

Example The Riemann zeta function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \dots = e^{-s \log(1)} + e^{-s \log(2)} + \dots$$

for $s \in \mathbb{C}$ with $\operatorname{Re} s > 1$.

Trigonometric functions From, for $y \in \mathbb{R}$

$$e^{iy} = \cos y + i \sin y,$$

can recover $\cos$ & $\sin$ via

$$\begin{aligned} \cos y &= \frac{e^{iy} + e^{-iy}}{2} = \frac{\cos y + i \sin y + \cos(-y) + i \sin(-y)}{2} \\ &= \frac{\cos y + \cos y + i(\sin y - \sin y)}{2} = \cos y, \end{aligned}$$

& similarly

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}.$$

Define, for $z \in \mathbb{C}$, $\sin(z)$ & $\cos(z)$ via

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

Define $\tan(z) = \frac{\sin(z)}{\cos(z)}$ if $\cos(z) \neq 0$.

Facts (More or less) all identities valid for $\sin(x), \cos(x), \tan(x)$ with x real, remain so for $\sin(z), \cos(z), \tan(z)$:

- $\sin(-z) = -\sin(z), \quad \cos(z) = \cos(-z)$
- $\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2) \quad \cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2)$
- $(\sin(z))^2 + \cos(z)^2 = 1$
- $\sin(z) = \sin(z + 2\pi), \quad \cos(z) = \cos(z + 2\pi)$

Each of these can be shown by inserting def'n of $\sin, \cos$ & calculating:

E.g.

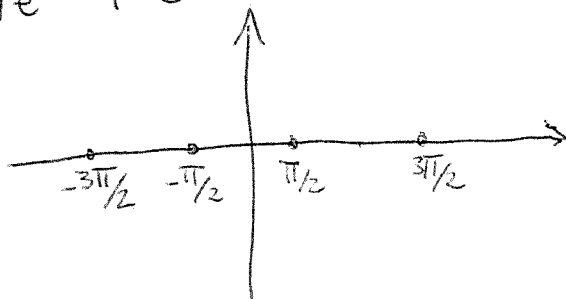
$$(\sin(z))^2 + (\cos(z))^2 = \left(\frac{e^{iz} - e^{-iz}}{2i} \right)^2 + \left(\frac{e^{iz} + e^{-iz}}{2} \right)^2 = \frac{e^{2iz} - 2e^{iz}e^{-iz} + e^{-2iz}}{-4} + \frac{e^{2iz} + 2e^{iz}e^{-iz} + e^{-2iz}}{4} = \frac{2}{4} + \frac{2}{4} = 1.$$

Later: "Principle of permanence of functional equations" explains this...

Where is $\cos(z) = 0$? $\cos(z) = \cos(x + iy) = \frac{e^{ix-y} + e^{-ix+y}}{2} = 0$

$$|e^{ix-y}| = e^{-y}$$

$$|e^{-ix+y}| = e^y$$



$$\begin{aligned} & \updownarrow \\ e^{ix-y} &= -e^{-ix+y} \\ & \updownarrow \\ y=0, & \quad e^{ix} = -e^{-ix} \\ & \updownarrow \\ e^{2ix} &= -1 \\ & \updownarrow \\ 2x &\in \pi + 2\pi\mathbb{Z} \\ x &\in \pi/2 + \pi\mathbb{Z}. \end{aligned}$$