

II.2 - Analytic functions

Let $U \subseteq \mathbb{C}$ be open, let $f: U \rightarrow \mathbb{C}$ be a function.

Def f is differentiable at $z_0 \in \mathbb{C}$ if the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The complex derivative of f at z_0 is then

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

~~Un~~unpacking this, f is differentiable if there exists a $f'(z_0) \in \mathbb{C}$ such that, for every $\delta > 0$, there exists an $\varepsilon > 0$ s.t.

$$|z - z_0| < \varepsilon \implies \left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \delta$$



Equivalently: For every sequence $z_1, z_2, \dots$ with $\lim z_i = z_0$,

$$\text{have } \lim_{n \rightarrow \infty} \frac{f(z_n) - f(z_0)}{z_n - z_0} = f'(z_0).$$

Examples: The constant function $f(z) = c$, $c \in \mathbb{C}$.

Then

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \rightarrow z_0} \frac{c - c}{z - z_0} = \lim_{z \rightarrow z_0} 0 = 0,$$

so $f'(z_0) = 0$ for all $z_0 \in \mathbb{C}$.

• The identity function $f(z) = z$,

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \rightarrow z_0} \frac{z - z_0}{z - z_0} = \lim_{z \rightarrow z_0} 1 = 1,$$

so $f(z)$ differentiable everywhere, $f'(z_0) = 1$.

Note Often useful to set $z = z_0 + \Delta z$, so

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

Ex Let $f(z) = \bar{z}$. Then

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{\overline{(z_0 + \Delta z)} - \bar{z}_0}{\Delta z} = \frac{\bar{z}_0 + \overline{\Delta z} - \bar{z}_0}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z}.$$

Does $\lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z}$ exist? Set $\Delta z = r e^{i\theta}$, then

$$\overline{\Delta z} = r e^{-i\theta}, \text{ so } \frac{\overline{\Delta z}}{\Delta z} = \frac{r e^{-i\theta}}{r e^{i\theta}} = e^{-2i\theta}.$$

$$\leadsto \frac{\overline{\Delta z}}{\Delta z} = e^{-2i \arg(\Delta z)},$$

• depends only on $\arg \Delta z$

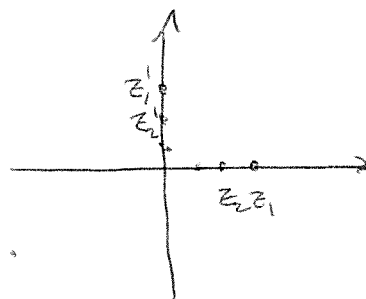
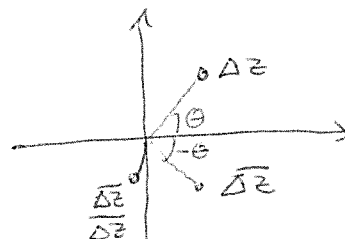
• varies with $\arg \Delta z$

$\Rightarrow$ limits along different rays from 0 will differ

$\Rightarrow \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z}$ does not exist.

E.g. $z_n = \frac{1}{n}$, gives $\lim_{n \rightarrow \infty} \frac{\bar{z}_n}{z_n} = 1$

$z_n = \frac{1}{n} i$, gives $\lim_{n \rightarrow \infty} \frac{\bar{z}_n}{z_n} = \frac{-1/n i}{1/n i} = \lim_{n \rightarrow \infty} -1 = -1.$



Ex Let $f(z) = z^m$. Then

$$\begin{aligned} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \frac{(z_0 + \Delta z)^m - z_0^m}{\Delta z} = \frac{z_0^m + m z_0^{m-1} \Delta z + \binom{m}{2} z_0^{m-2} \Delta z^2 + \dots - z_0^m}{\Delta z} \\ &= \frac{\sum_{i=1}^m \binom{m}{i} z_0^{m-i} \Delta z^i}{\Delta z} = m z_0^{m-1} + \Delta z (\dots) \end{aligned}$$

As $\Delta z \rightarrow 0$, this tends to $m z_0^{m-1}$, so $f(z)$ is differentiable at all points, with $f'(z) = m z^{m-1}$

Thm If $f(z)$ is differentiable at $z_0 \in \mathbb{C}$, then it is continuous at z_0 .

Pf Must show, given $\epsilon > 0$, $\exists \delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \epsilon$.

~~$$\text{Kudun, EUSIAH } \frac{f(z) - f(z_0)}{z - z_0} \rightarrow f'(z_0) \text{ if } |z - z_0| \rightarrow 0$$~~

for any sequence $z_n \rightarrow z_0$, that $f(z_n) \rightarrow f(z_0)$. Differentiable implies

$$\frac{f(z_n) - f(z_0)}{z_n - z_0} \rightarrow f'(z), \quad \& \quad |z_n - z_0| \rightarrow 0$$

$$\begin{aligned} f(z_n) - f(z_0) &= (z_n - z_0) \frac{f(z_n) - f(z_0)}{(z_n - z_0)} \rightarrow 0 \cdot f'(z) = 0 \\ &\Downarrow \\ f(z_n) &\rightarrow f(z_0). \end{aligned}$$

E.g. $f(z) = \begin{cases} z & \text{if } z \neq 0 \\ 1 & \text{if } z = 0 \end{cases}$ not differentiable at 0.

Thm (All usual ~~derivation~~ derivation rules apply). Let $f(z), g(z)$ be complex functions, and assume both are differentiable at z . Then

① $(cf)'(z) = cf'(z)$

② ~~$f(z)g(z)$~~ $(f+g)'(z) = f'(z) + g'(z)$

③ $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

④ $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$

that is, all four functions are differentiable at z with derivatives as given.

Pf Proved as in real analysis, e.g. for 3. Assume $\Delta z \rightarrow 0$, then

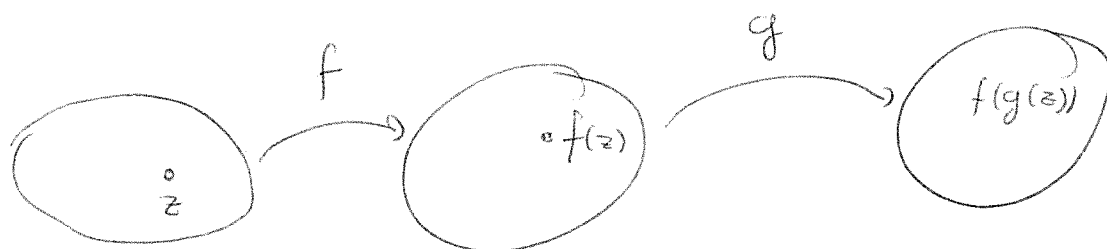
$$\begin{aligned} \frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z)}{\Delta z} &= \frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z)}{\Delta z} \\ &= \frac{(f(z+\Delta z) - f(z))g(z) + f(z+\Delta z)(g(z+\Delta z) - g(z))}{\Delta z} \rightarrow f'(z)g(z) + f(z)g'(z) \end{aligned}$$

Ex If $f(z) = c_n z^n + c_{n-1} z^{n-1} + \dots + c_0$, then $f(z)$ differentiable everywhere &

$$f'(z) = c_n n z^{n-1} + c_{n-1} (n-1) z^{n-2} + \dots + c_1.$$

Thm (Chain rule) If $f(z)$, $g(z)$ are functions, with $f(z)$ differentiable at z & g differentiable at $f(z)$, then $g \circ f$ is differentiable at z , and

$$(g \circ f)'(z) = g'(f(z)) f'(z).$$



Pf ~~where~~ Case I: $f'(z) \neq 0$. Then for $|\Delta z|$ small, have

$$\left| \frac{f(z+\Delta z) - f(z)}{\Delta z} - f'(z) \right| < \frac{1}{2} |f'(z)|$$

$$\Downarrow$$

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} \neq 0$$

$$\Downarrow$$

$$f(z+\Delta z) - f(z) \neq 0.$$

$$\text{f''(z)}$$

$$\text{Then } \frac{g(f(z+\Delta z)) - g(f(z))}{\Delta z} = \frac{g(f(z+\Delta z)) - g(f(z))}{f(z+\Delta z) - f(z)} \cdot \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

$$\rightarrow g'(f(z)) f'(z), \text{ as } \Delta z \rightarrow 0$$

Case II: $f'(z) = 0$. Let $w = f(z)$, $\Delta w = f(z+\Delta z) - f(z)$. ~~For~~ For small $|\Delta w|$, have

$$\left| \frac{g(w+\Delta w) - g(w)}{\Delta w} \right| < 2|g'(w)| =: C$$

$$\Rightarrow |g(w+\Delta w) - g(w)| < C|\Delta w|$$

Then $\left| \frac{g(w+\Delta w) - g(w)}{\Delta z} \right| \leq \left| \frac{C \Delta w}{\Delta z} \right|$, and since $\frac{\Delta w}{\Delta z} \rightarrow 0$ as $\Delta z \rightarrow 0$, have $\frac{g(w+\Delta w) - g(w)}{\Delta z} \rightarrow 0 = g'(f(z))f'(z)$.

Def A complex function f defined on an open set $U \subseteq \mathbb{C}$ is analytic or holomorphic if it is differentiable at all points of U , & the function $f'(z)$ is continuous. □

Ex $f(z) = c_n z^n + \dots + c_0$ is analytic on $\mathbb{C}$.

Ex $\bar{z}$ is nowhere analytic

• $|z|^2$ is differentiable at 0 , but nowhere else, so not analytic on any open set.

Remark Condition " $f'(z)$ continuous" follows from

" $f'(z)$ exists" (Goursat's theorem). We will not prove this, but therefore not worry about checking continuity of $f'(z)$.

Ex Is e^z analytic? Next time...