

## II.3 - II.4: Cauchy-Riemann equations & inverse mapping theorem.

Recall function  $f(z)$  on  $U \subseteq \mathbb{C}$  is differentiable at  $z \in U$  if

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

exists, & analytic if it is differentiable at all points, with  $f'(z)$  continuous.

Write  $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ ,  $u, v$  real, & assume  $f$  is analytic. Can compute  $f'(z)$  by taking limit in 2 ways:

$$\begin{aligned} \textcircled{1} \quad \Delta z \text{ real, } \Delta z = \Delta x \quad f'(z) &= \lim_{\Delta x \rightarrow 0} \frac{f(z+\Delta x) - f(z)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{u(x+\Delta x, y) + iv(x+\Delta x, y) - u(x, y) - iv(x, y)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \left( \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x} \right) \\ &= u_x(x, y) + i v_x(x, y), \end{aligned}$$

with  $u_x, v_x$  denoting partial derivatives.

$$\begin{aligned} \textcircled{2} \quad \Delta z \text{ imaginary, } \Delta z = i\Delta y, \\ f'(z) &= \lim_{\Delta y \rightarrow 0} \frac{f(z+i\Delta y) - f(z)}{i\Delta y} = \lim_{\Delta y \rightarrow 0} \left( \frac{u(x, y+\Delta y) - u(x, y)}{i\Delta y} + i \frac{v(x, y+\Delta y) - v(x, y)}{i\Delta y} \right) \\ &= v_y(x, y) - i u_y(x, y). \end{aligned}$$

In both cases, existence of limit implies partial derivatives of  $u, v$  exist,  $f'(z)$  continuous  $\Rightarrow u_x, u_y, v_x, v_y$  continuous. Also get:

$$u_x(x, y) + i v_x(x, y) = v_y(x, y) - i u_y(x, y)$$

$$\begin{array}{l} \text{Cauchy-Riemann:} \\ \text{equations} \end{array} \quad \begin{array}{l} \text{I} \quad u_x(x, y) = v_y(x, y) \\ \text{II} \quad u_y(x, y) = -v_x(x, y). \end{array}$$

Def A function  $[u(x,y), v(x,y)]$  is  $C^1$  if  $u_x, u_y, v_x, v_y$  exist and are continuous everywhere.

Thm A function  $f: U \rightarrow \mathbb{C}$ ,  $f(z) \Rightarrow u(x,y) + iv(x,y)$  is analytic if and only if it is  $C^1$  & CR equations hold.

Pf  $f$  analytic  $\Rightarrow f$   $C^1$  & CR eqns hold done above

For  $\Leftarrow$ , must show  $\lim_{\Delta z \rightarrow 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$  exists & is continuous. ( $C^1$ !)

Let  $\Delta z = \Delta x + i\Delta y$ ,  $f(z) = u(x,y) + iv(x,y)$ , Taylor expansion is

$$u(x+\Delta x, y+\Delta y) = u(x,y) + \Delta x u_x(x,y) + \Delta y u_y(x,y) + \epsilon_u(\Delta x, \Delta y) |\Delta z|$$

$$v(x+\Delta x, y+\Delta y) = v(x,y) + \Delta x v_x(x,y) + \Delta y v_y(x,y) + \epsilon_v(\Delta x, \Delta y) |\Delta z|,$$

with  $\epsilon_u, \epsilon_v \rightarrow 0$  as  $\Delta z \rightarrow 0$ . Then (\*) Taylor exp. + CR eqns

$$\frac{f(z+\Delta z) - f(z)}{\Delta z} \stackrel{(*)}{=} \frac{(\Delta x + i\Delta y)(u_x(x,y) + i v_x(x,y)) + (\epsilon_u + i\epsilon_v) |\Delta z|}{\Delta z}$$

$$= u_x(x,y) + i v_x(x,y) + (\epsilon_u + i\epsilon_v) \frac{|\Delta z|}{\Delta z}$$

As  $\Delta z \rightarrow 0$ , last term  $\rightarrow 0$ , so  $f'(z) = u_x(x,y) + i v_x(x,y)$ , & this is continuous by  $C^1$  assumption.

Ex  $f(z) = z \iff u(x,y) = x \quad v(x,y) = y$   
 $\quad \quad \quad = x + iy$

$$u_x = 1 \quad v_x = 0$$

$$u_y = 0 \quad v_y = 1$$

$$u_x = v_y \quad \text{holds.}$$

$$u_y = -v_x$$

$f$  analytic

$$f(z) = \bar{z} = x - iy, \quad u = x, \quad v = -y$$

$$u_x = 1 \quad v_x = 0$$

$$u_y = 0 \quad v_y = -1$$

$$u_x = -v_y \text{ fails}$$

$f$  not analytic

Ex  $f(z) = e^z = e^{x+iy} = e^x (\cos y + i \sin y)$ .

$$u = e^x \cos y \quad u_x = e^x \cos y \quad u_y = -e^x \sin y$$

$$v = e^x \sin y \quad v_x = e^x \sin y \quad v_y = e^x \cos y$$

$$u_x = v_y \quad u_y = -v_x \implies e^z \text{ is analytic.}$$

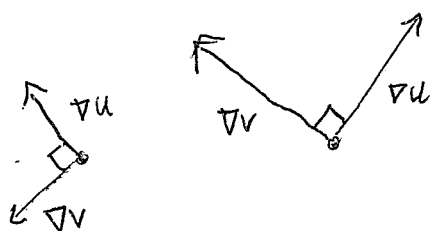
Cor  $\cos z = \frac{e^{iz} + e^{-iz}}{2}$ ,  $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$  analytic.

"Geometric interpretation" of CR equations.

$f(z) = u(x,y) + iv(x,y)$ . At a point  $(x,y)$

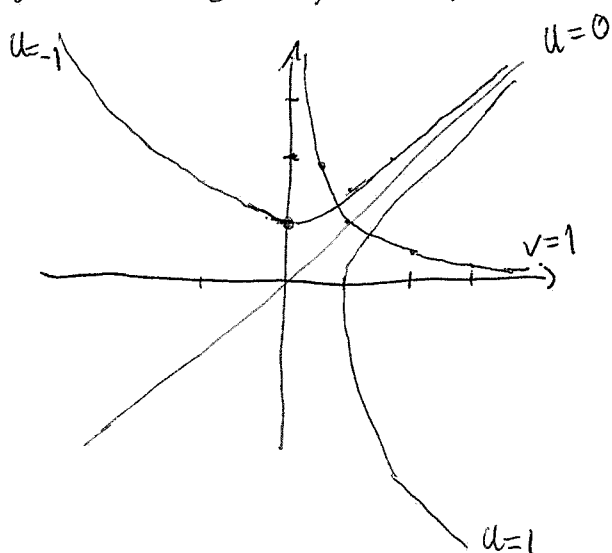
$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \nabla u \quad \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \nabla v$$

CR eqns:  $\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} v_y \\ -v_x \end{pmatrix} = \nabla v \text{ rotated } \pi/2 \text{ clockwise}$



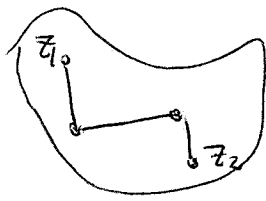
Level ~~of~~ curve of  $u$  is normal to  $\nabla u$ , level curve of  $v$  is normal to  $\nabla v \implies$  level curves of  $u$  &  $v$  cross normally

$f(z) = z^2 = (x^2 - y^2) + i2xy$

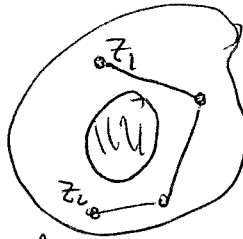


~~level~~

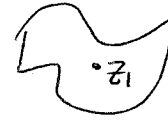
Recall an open  $U \subseteq \mathbb{C}$  is a domain if it is connected, meaning any two  $z_1, z_2$  can be joined by a chain of line segments in  $U$



domain



domain



not domain.

Fact: If  $D$  is domain &  $h: D \rightarrow \mathbb{R}$  is  $C^1$  with  $h_x = h_y \equiv 0$ , then  $h$  is constant.

Thm Let  $f(z)$  be analytic on a domain  $D$ , & assume  $f'(z) \equiv 0$ . Then  $f(z)$  is constant.

Pf  $f = u + iv$ ,  $0 = f'(z) = u_x(x, y) + i v_x(x, y) = v_y(x, y) - i u_y(x, y)$   
 $\Rightarrow u_x, u_y, v_x, v_y \equiv 0 \Rightarrow u \text{ \& } v \text{ constant} \Rightarrow f \text{ constant}$

Thm Let  $f(z)$  be analytic on a domain  $D$ , & assume  $f(z) \in \mathbb{R}$  for all  $z \in D$ . Then  $f$  is constant.

Pf  $f = u + iv$ .  $f(z) \in \mathbb{R} \Rightarrow v \equiv 0$ ,  $u_x = v_y \equiv 0$ ,  $u_y = -v_x \equiv 0 \Rightarrow u \text{ constant}$   
 $\Rightarrow f \text{ constant}$

Ex  $|z|$ , ~~any~~  $\text{Arg } z$  not analytic.

The Jacobian matrix & inverse mapping theorem.

~~Recall for  $U \subseteq \mathbb{R}^n$   $f: U \rightarrow \mathbb{R}^n$  is a diffeomorphism if  $f$  is a  $C^1$  bijection with  $C^1$  inverse.~~

For a  $C^1$  function  $f(z) = u(x, y) + iv(x, y)$ , the Jacobian matrix is

$$Jf = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

Ex  $f(z) = z^2 = (x^2 - y^2) + i 2xy$

$$Jf = \begin{pmatrix} 2x & 2y \\ -2y & 2x \end{pmatrix}$$

Note If  $f'(z)$  is analytic, then

$$\det Jf = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \stackrel{\text{CR eqns}}{=} \det \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix} \\ = u_x^2 + v_x^2 = |u_x + i v_x|^2 = |f'(z)|^2$$

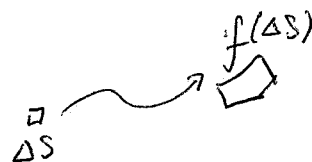
Cor If  $f'(z) \neq 0$ , then  $Jf$  is invertible (& vice versa).

Note  $|\det Jf(z)|$  measures area of  $f(\Delta S)$  for "infinitesimal square"  $\Delta S$  at  $z$

$$\text{Area } f(\Delta S) = |\det Jf(z)| \text{ Area } \Delta S.$$

If  $f$  is analytic & injective on  $U$ , then

$$\text{Area } f(U) = \iint_U |f'(z)|^2 dx dy.$$



Inverse mapping theorem Let  $f(z)$  be analytic, on domain  $D$ , & assume  $z_0 \in D$  is such that  $f'(z_0) \neq 0$ . Then on some open  $U \subseteq D$  containing  $z_0$ ,

(1) ~~there exists a unique~~  $f^{-1}$  such that  $f^{-1} \circ f = \text{id}_U$

$U \subseteq D$  with  $z_0 \in U$ ,  $f$  is injective,  $V = f(U) \subseteq \mathbb{C}$  is open,

&  $f^{-1}: V \rightarrow U$  is analytic, with

$$(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}.$$

Pf  $f'(z_0) \neq 0 \Rightarrow Jf$  invertible; real inverse mapping thm gives  $U$ ,  $f$  injective,  $f(U)$  open, and  $f^{-1}$  is  $C^1$ .

~~Need to show~~

Need to show  $f^{-1}$  analytic.

Observe:  $J_f = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$ , CR eqns  $\Leftrightarrow J_f$  has form  $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$

Chain rule:  $J_{f^{-1}} \circ J_f = J_{\text{id}} = I_2 \Rightarrow J_{f^{-1}} = J_f^{-1} = \frac{1}{\det J_f} \begin{pmatrix} A & B \\ B & A \end{pmatrix}$

If  $J_f = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ , then  $(J_f)^{-1} = \frac{1}{\det J_f} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} = \begin{pmatrix} C & D \\ -D & C \end{pmatrix}$

$C = \frac{A}{\det J_f}$ ,  $D = \frac{-B}{\det J_f}$ , so CR eqns hold for  $J_{f^{-1}} = (J_f)^{-1}$ ,  $f^{-1}$  analytic

Finally chain rule gives

~~1 = id'(z)~~  $1 = \text{id}'(z) = (f \circ f^{-1})'(z) = f'(f^{-1}(z)) \cdot (f^{-1})'(z)$ , so  
 $(f^{-1})'(z) = \frac{1}{f'(f^{-1}(z))}$

Ex Let  $f(z) = e^z$ ,  $z_0 \in \mathbb{C}$ .  $f'(z_0) = e^{z_0} \neq 0$ .

Near  $e^{z_0}$ , can define an inverse, i.e.  $g(z)$  s.t.

$e^{g(z)} = z \leadsto g(z)$  a branch of  $\log(z)$ ,

with  $g(z)$  analytic, and

$$g'(z) = \frac{1}{\cancel{e^{g(z)}} f'(g(z))} = \frac{1}{e^{g(z)}} = \frac{1}{z}.$$

E.g.  $\text{Log}(z): \mathbb{C} \setminus \mathbb{R}_{\leq 0} \rightarrow \mathbb{C}$  is analytic with  $\text{Log}'(z) = \frac{1}{z}$ .

Ex  $f(z) = z^n$ ,  $z_0 \in \mathbb{C} \setminus \{0\}$ . Then  $f'(z_0) = n z_0^{n-1} \neq 0$ , so near

$f(z_0)$ , can find inverse  $g(z)$  (branch of  $\sqrt[n]{z}$ ), <sup>analytic</sup> with  $g(z)^n = z$

$$g'(z) = \frac{1}{f'(g(z))} = \frac{1}{n g(z)^{n-1}} = \frac{g(z)}{n z}.$$