

II.7 Fractional Linear Transformations

Def A fractional linear transformation (FLT) or Möbius transformation is a function of the form

$$f(z) = \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{C},$$

where $ad-bc \neq 0$.

Note If $ad-bc=0$, then: $c=d=0 \leadsto \frac{az+b}{cz+d}$ undefined
 $c \neq 0$ or $d \neq 0$:

$$\left(\frac{az+b}{cz+d}\right)' = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2} = 0,$$

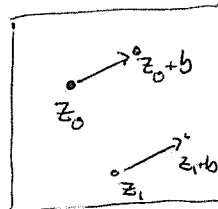
Examples ~~ALL~~ Special kinds of FLTs:

• $f(z) = \frac{1z+b}{0z+1} = z+b$, translation

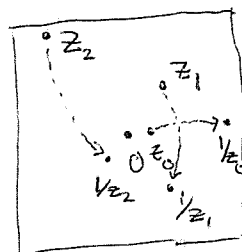
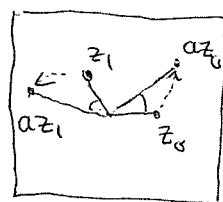
• $f(z) = \frac{az+0}{0z+1} = az$ ($a \neq 0$), dilation

• $f(z) = \frac{0z+1}{1z+0} = \frac{1}{z}$, inversion

Translation



Dilation



An FLT $\frac{az+b}{cz+d}$ is well-defined for z s.t. $cz+d \neq 0$. It is useful to extend definition of it to extended complex plane $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$.

Def An FLT $\frac{az+b}{cz+d}$ is called an affine transformation if $c=0$, equivalently if it has the form $f(z) = az+b$.

Def For an FLT $f(z) = \frac{az+b}{cz+d}$, we define its extension $f: \mathbb{C}^* \rightarrow \mathbb{C}^*$ by

• If f is affine transformation, $f(\infty) = \infty$.

• If f is not affine transformation, so $c \neq 0$, define

$$f\left(-\frac{d}{c}\right) = \infty, \quad f(\infty) = \frac{a}{c}$$

Note These extensions are reasonable, since in the affine case

$$\lim_{z \rightarrow \infty} (az+b) = \infty,$$

in the non-affine case

$$\lim_{z \rightarrow -\frac{d}{c}} \frac{az+b}{cz+d} = \infty, \quad \lim_{z \rightarrow \infty} \frac{az+b}{cz+d} = \frac{a}{c} \quad \text{by } \lim_{z \rightarrow \infty} \frac{a + \frac{b}{z}}{c + \frac{d}{z}} = \frac{a+0}{c+0} = \frac{a}{c}.$$

Def If $f(z)$ is a complex function, ~~we say~~ $z_0 \in \mathbb{C}$

• $\lim_{z \rightarrow z_0} f(z) = \infty$ if for all $N > 0$, there is a $\delta > 0$ s.t. $|z - z_0| < \delta \Rightarrow |f(z)| > N$.

• $\lim_{z \rightarrow \infty} f(z) = z_0$ if for all $\varepsilon > 0$, there is an $N > 0$ s.t. $|z| > N \Rightarrow |f(z) - z_0| < \varepsilon$.

• $\lim_{z \rightarrow \infty} f(z) = \infty$ if for all $M > 0$, there is an N s.t. $|z| > N \Rightarrow |f(z)| > M$.

Ex $f(z) = \frac{z+2}{2z+i}$. Have $f(z) \in \mathbb{C}$ if $2z+i \neq 0 \Rightarrow$ if $z \neq -\frac{i}{2}$.
 $f(-\frac{i}{2}) = \infty$, and $f(\infty) = \lim_{z \rightarrow \infty} f(z) = \frac{1}{2}$.

Note $f(z) = \frac{az+b}{cz+d}$ does not uniquely determine a, b, c, d , since,
 for $\lambda \in \mathbb{C} \setminus \{0\}$, $\frac{\lambda a z + \lambda b}{\lambda c z + \lambda d} = \frac{az+b}{cz+d}$, $(\lambda a, \lambda b, \lambda c, \lambda d)$ gives same FLT as (a, b, c, d) .

Thm As a map $f: \mathbb{C}^* \rightarrow \mathbb{C}^*$, an FLT is a bijection, and the inverse $f^{-1}: \mathbb{C}^* \rightarrow \mathbb{C}^*$ is an FLT

Pf: Let $w \in \mathbb{C}$, $f(z) = \frac{az+b}{cz+d}$. If $z \in \mathbb{C}$ is s.t. $w = f(z)$, have
 $w = \frac{az+b}{cz+d} \Leftrightarrow w(cz+d) = az+b \Leftrightarrow (a-wc)z = wd-b$
 $\Rightarrow z = \frac{wd-b}{-wc+a}$

$f^{-1}(w) = \frac{wd-b}{-wc+a}$ is an inverse to $f(z)$.

Thm Let $f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}$, $f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$ be FLTs. Then

$$f_1 \circ f_2(z) = f_1(f_2(z)) \text{ is an FLT}$$

$$\begin{aligned} \text{Pf } f_1(f_2(z)) &= \frac{a_1 \left(\frac{a_2 z + b_2}{c_2 z + d_2} \right) + b_1}{c_1 \left(\frac{a_2 z + b_2}{c_2 z + d_2} \right) + d_1} = \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} \\ &= \frac{(a_1 a_2 + b_1 c_2)z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2)z + (c_1 b_2 + d_1 d_2)} = \frac{a_3 z + b_3}{c_3 z + d_3} \end{aligned}$$

Must check $a_3 d_3 - b_3 c_3 \neq 0$.

$$\bullet ad - bc = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\bullet \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$$\begin{aligned} \bullet \det \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} &= \det \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \det \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ &= (\neq 0) \cdot (\neq 0) = (\neq 0). \end{aligned}$$

Cor Composition of FLTs computable by matrix product:

$$f(z) = \frac{z+2}{z+1} \quad g(z) = \frac{1}{z+1} \quad f \circ g(z) = \frac{2z+3}{z+2}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

Formally: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \frac{az+b}{cz+d}$ is a homomorphism from the group $GL(2, \mathbb{C})$ of invertible matrices to the group of FLTs with composition as group product.

Defining FLTs by specifying values.

Thm Let $z_1, z_2, z_3 \in \mathbb{C}^*$ be distinct, let $w_1, w_2, w_3 \in \mathbb{C}^*$ be distinct. There is a unique FLT f s.t.

$$f(z_1) = w_1$$

$$f(z_2) = w_2$$

$$f(z_3) = w_3$$

Pf Consider first special case $w_1 = 0, w_2 = 1, w_3 = \infty$.

Assume for simplicity $z_1, z_2, z_3 \neq 0$. Then

$$f(z) = \frac{z - z_1}{z - z_3} \cdot \frac{z_2 - z_3}{z_2 - z_1} \text{ is an FLT s.t.}$$

$$f(z_1) = 0$$

$$f(z_2) = 1$$

$$f(z_3) = \infty.$$

If g is another FLT s.t. $g(z_1) = 0, g(z_2) = 1, g(z_3) = \infty$, then $f \circ g^{-1}$ is an FLT s.t. $(f \circ g^{-1})(0) = 0, (f \circ g^{-1})(1) = 1, (f \circ g^{-1})(\infty) = \infty$.

Since $(f \circ g^{-1})(\infty) = \infty$, $f \circ g^{-1}$ is affine, so $f \circ g^{-1} = az + b$.

$$f \circ g^{-1}(0) = 0 \Rightarrow b = 0$$

$$f \circ g^{-1}(1) = 1 \Rightarrow a = 1.$$

$$\text{So } f \circ g^{-1}(z) = z \Rightarrow f = g.$$

Now for general case: Given $w_1, w_2, w_3, z_1, z_2, z_3$, let f, g be

$$\text{FLT's s.t. } f(w_1) = 0, f(w_2) = 1, f(w_3) = \infty$$

$$g(z_1) = 0, g(z_2) = 1, g(z_3) = \infty.$$

Then $f \circ g^{-1}$ maps z_1, z_2, z_3 to w_1, w_2, w_3 .

If h is an FLT with the same property, then

$$f \circ h \circ g^{-1} \text{ fixes } 0, 1, \infty, \text{ so } f \circ h \circ g^{-1} = \text{id}_{\mathbb{C}^*} \Rightarrow h = f^{-1} \circ g.$$

Decomposing FLTs

Thm Let f be an FLT. Then f can be written as a composition of dilations, translations & inversions

Pf Case I: $f(\infty) = \infty \iff f$ affine transformation

$$\iff f(z) = az + b, \quad a \neq 0$$

$$\iff f = (z+b) \circ (az).$$

Case II: $f(\infty) = w \neq \infty$. Let $g = \frac{1}{z} \circ (z-w) \circ f$, then $g(\infty) = \infty$,

so can write $g(z) = (z+b) \circ (az)$. Thus

$$(z+b) \circ (az) = \frac{1}{z} \circ (z-w) \circ f$$

$\Downarrow$

$$(z-w)^{-1} \circ \left(\frac{1}{z}\right)^{-1} \circ (z+b) \circ (az) = f$$

$$(z+w) \circ \frac{1}{z} \circ (z-b) \circ az = f.$$

Thm ~~Thm~~