

Recall a circle in $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$ is a circle on the Riemann sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Under stereographic projection, this corresponds to two kinds of sets

- Circles in $\mathbb{C}$
- Lines in $\mathbb{C}$, together with ∞ .

~~The~~ "A circle through ∞ is a line".

Thm Let $f(z) = \frac{az+b}{cz+d}$ be a fractional linear transformation. Then f maps ~~every~~ every circle in $\mathbb{C}^*$ to a circle in $\mathbb{C}^*$.

Pf Recall f can be written as a composition of three kinds of FLTs

- Translation: $z \mapsto z+b$
- Dilation: $z \mapsto az$ ($a \neq 0$)
- Inversion: $z \mapsto \frac{1}{z}$

It is easy to see the first two map circles to circles, e.g. under $z \mapsto az$, circle $\{z \mid |z-z_0|=r\}$ maps to $\{z \mid |z-az_0|=|a|r\}$. Need only

Lemma Inversion maps circles in $\mathbb{C}^*$ to circles in $\mathbb{C}^*$.

Pf We claim that under stereographic projection

$z \mapsto \frac{1}{z}$ corresponds to rotation by π around x-axis

$(a, b, c) \mapsto (a, -b, -c)$. To see this note st. projection is

$$(a, b, c) \mapsto \frac{a+ib}{1-c}$$

$$(a, b, -c) \mapsto \frac{a-ib}{1+c} = (1-c) \frac{a-ib}{1-c^2} = (1-c) \frac{a-ib}{a^2+b^2} = (1-c)(a+ib)^{-1} = \left(\frac{a+ib}{1-c} \right)^{-1}$$

Clearly rotation by π maps circles to circles, so the proof is complete $\square$

Example (Exam 2024) Let $f(z) = \frac{-z+1}{2z+3}$. Compute inverse to $f(z)$, and find the image of $\{z \mid |z-1|=2\}$ under f .

Inverse: $f \leftrightarrow \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}$, inverse matrix $= \frac{1}{\det \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix}} \begin{pmatrix} 3 & -1 \\ -2 & -1 \end{pmatrix}$
 $f^{-1}(z) = \frac{3z-1}{-2z-1}$.

Let $S = \{z \mid |z-1|=2\}$.

Compute $f(3) = \frac{2}{9}$, $f(-1) = 2$

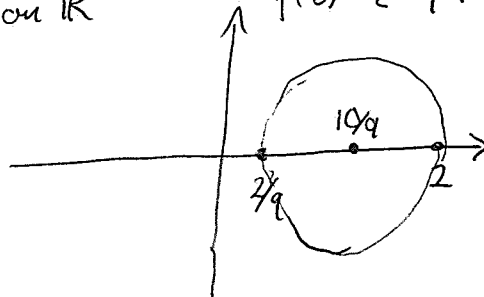
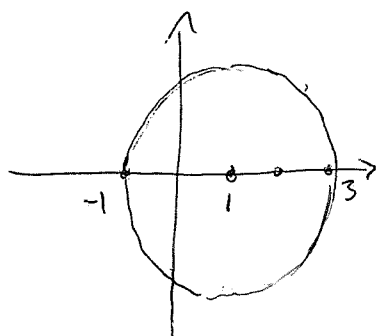
~~$f(x) \in \mathbb{R} \Rightarrow f(x) \in \mathbb{R}$~~ , so $f(\mathbb{R}) = \mathbb{R}$.

S & $\mathbb{R}$ meet orthogonally, so $f(S)$ meets

$f(\mathbb{R}) = \mathbb{R}$ orthogonally $\Rightarrow$ centre of $f(S)$ is on $\mathbb{R}$

$f(S) = \{z \mid |z - \frac{10}{9}| = \frac{8}{9}\}$

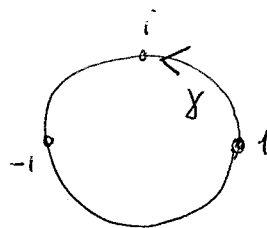
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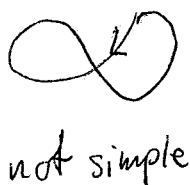
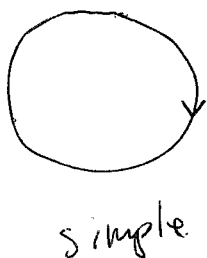
III.1 Line integrals

A path in $\mathbb{C}$ is a continuous function $\gamma: [a, b] \rightarrow \mathbb{C}$, where $[a, b] \subseteq \mathbb{R}$.

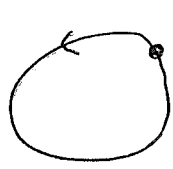
E.g. $\gamma: [0, 2\pi] \rightarrow \mathbb{C}$, $\gamma(t) = e^{it}$



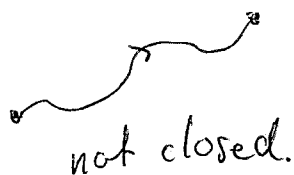
We say a path is simple if for all $a < t_0 < t_1 < b$, $\gamma(t_0) \neq \gamma(t_1)$.



Say γ is closed if $\gamma(a) = \gamma(b)$



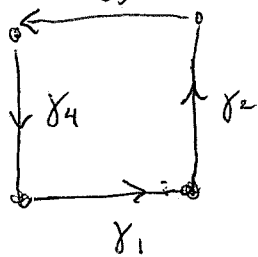
closed



not closed.

Say $\gamma(t) = x(t) + iy(t)$ is C^k if x, y are k times continuously differentiable, and say $\gamma(t)$ is "smooth" if it is C^k for whatever k the application calls for.

Say γ is piecewise smooth if it is the concatenation of smooth paths.



Def A differential on an open set $U \subseteq \mathbb{C}$ is a symbol

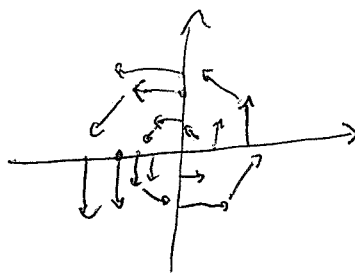
$$P(x, y) dx + Q(x, y) dy,$$

where P, Q are continuous functions on U .

Note P, Q can be real- or complex-valued.

If P, Q are real, we may think of and visualise $Pdx + Qdy$ as the vector field $(P(x, y), Q(x, y))$, e.g.

$$-y dx + x dy \longleftrightarrow (-y, x)$$



Note Differentials are not really

vector fields, since they behave differently under variable change.

Usually harmless to pretend they are...

Differentials are things which can be integrated along paths.

Def Let γ be a smooth path, and let $\omega = P(x,y)dx + Q(x,y)$ a differential.

Then

$$\int_{\gamma} \omega = \int_{\gamma} Pdx + Qdy = \int_a^b (P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)) dt.$$

Note May also define $\int_{\gamma} \omega$ as Riemann sum, given $a = t_0 < t_1 < \dots < t_n = b$

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} (P(x(t_i), y(t_i))x'(t_i) + Q(x(t_i), y(t_i))y'(t_i)) \Delta t_i$$

have $\sum_{i=0}^{n-1} (P(x(t_i), y(t_i))x'(t_i) + Q(x(t_i), y(t_i))y'(t_i)) \Delta t_i$

Note Under correspondence with vector fields, $Pdx + Qdy \leftrightarrow (P, Q)$,

have

$$\int_{\gamma} \omega = \int_a^b \gamma'(t) \cdot (P, Q) dt.$$

Ex $\omega = -ydx + xdy$, $\gamma(t) = e^{it} = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$.

Then

$$\begin{aligned} \int_{\gamma} \omega &= \int_0^{2\pi} \sin(t)(\cos(t))' + \cos(t)(\sin(t))' dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt = \int_0^{2\pi} 1 dt. \end{aligned}$$

Green's theorem Let $U \subseteq \mathbb{C}$ be a domain. Say

U is bounded if there is some $R \geq 0$ s.t.

$|z| \leq R$ for all $z \in U$.

Bounded: Unit disc

Unbounded: $\mathbb{C}$, upper half-plane.

Def ~~Then~~ ~~ext~~ Let $\omega = P(x,y)dx + Q(x,y)dy$. The exterior derivative of ω is

$$d\omega = \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dx dy.$$

Note $-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = \text{curl}((P, Q)).$

~~Ex~~ Ex $d(-ydx + xdy) = (1+1)dx dy = 2dx dy$
known

Thm (Green's theorem) Let D be a domain with piecewise smooth, closed boundary. Let $Pdx + Qdy$ be a differential defined on some open set $U \supset D$. ~~Then~~ Let γ be a path tracing the boundary of D , ~~then~~ in a counterclockwise direction. Then

$$\int_{\gamma} Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy$$

$$\int_{\gamma} \omega = \iint_D d\omega.$$

Ex $\gamma = e^{it} \quad t \in [0, 2\pi] \quad D = \{z \mid |z| < 1\}, \quad \omega = -ydx + xdy.$

$$\int_{\gamma} \omega = \iint_D d\omega = \iint_D 2 dx dy = 2 \cdot \text{Area}(D) = 2\pi.$$