

III.2 - Independence of path

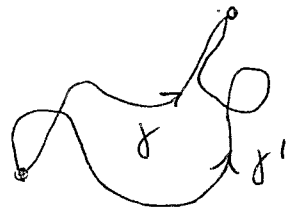
Consider open $U \subseteq \mathbb{C}$, differential $Pdx + Qdy$ on U

For $\gamma: [a, b] \rightarrow U$ a path, have $\int_{\gamma} Pdx + Qdy$, & ask:

Does $\int_{\gamma} Pdx + Qdy$ depend only on $\gamma(a), \gamma(b)$?

That is, if $\tilde{\gamma}: [c, d] \rightarrow U$ has $\tilde{\gamma}(c) = \gamma(a)$,

$\tilde{\gamma}(d) = \gamma(b)$, is $\int_{\tilde{\gamma}} Pdx + Qdy = \int_{\gamma} Pdx + Qdy$?



In general, integrals depend on choice of path, so we want conditions on U , $Pdx + Qdy$, γ to make integrals independent of path.

Recall for a real function $f(x)$, a function $F(x)$ is an antiderivative of f if $F' = f$.

Thm (Fundamental theorem of calculus)

- If F is antiderivative of f , then

$$\int_a^b f(x) dx = F(b) - F(a)$$

- If f is continuous, then

$$F(x) = \int_a^x f(t) dt$$

is an antiderivative.

Def Let $U \subseteq \mathbb{C}$, let $h(x, y)$ be a smooth (real or complex) function on U . The exterior derivative of h is the differential

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$$

Note ~~iff~~ Under correspondence differentials $\leftrightarrow$ vector fields

$$dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy \leftrightarrow \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right) = \nabla h, \text{ gradient.}$$

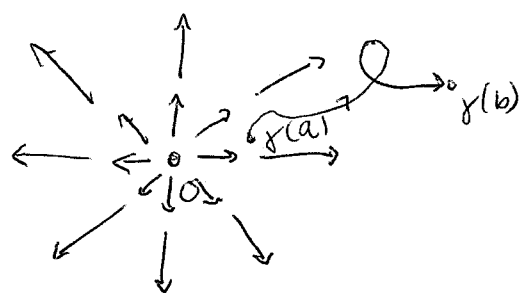
Def A differential ω is exact if there is a function h s.t. $\omega = dh$.

Thm Let $U \subseteq \mathbb{C}$ be open h a smooth function on U . Let $\omega = dh$, and let $\gamma: [a, b] \rightarrow U$ be a smooth path. Then

$$\int_{\gamma} \omega = h(\gamma(b)) - h(\gamma(a)).$$

$$\begin{aligned} \text{Pf } \int_{\gamma} \omega &= \int_{\gamma} \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy = \int_a^b \frac{\partial h}{\partial x}(\gamma(t)) x'(t) + \frac{\partial h}{\partial y}(\gamma(t)) y'(t) dt \\ &= \int_a^b \frac{d}{dt} h(x(t), y(t)) dt \stackrel{\text{FTC}}{=} h(\gamma(b)) - h(\gamma(a)). \end{aligned}$$

Ex $h(x, y) = x^2 + y^2, \quad \omega = 2x dx + 2y dy$



$$\begin{aligned} \int_{\gamma} \omega &= \int_a^b 2x(t)x'(t) dt + 2y(t)y'(t) dt \\ &= \int_a^b \frac{d}{dt} (x(t)^2 + y(t)^2) dt = x(b)^2 + y(b)^2 - x(a)^2 - y(a)^2 \end{aligned}$$

Thm Let $U \subseteq \mathbb{C}$ be ~~open~~ ^{a domain}, and let ω be a smooth differential. ~~such that~~ Then the following are equivalent

- ① ω is exact
- ② For any two ^{paths} γ_1, γ_2 ⁱⁿ U with the same start and end point, $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$
- ③ For any closed path γ in U , $\int_{\gamma} \omega = 0$.

Pf Have seen $(1) \Rightarrow (2)$.

$(2) \Rightarrow (3)$ Given closed path $\gamma: [a, b] \rightarrow U$ with $(x_0, y_0) = \gamma(a) = \gamma(b)$,

Let $\gamma_0: [a, b] \rightarrow U$ be constant path at (x_0, y_0)

$$\gamma_0(t) = (x_0, y_0).$$

$$\text{Then } \int_{\gamma} \omega = \int_{\gamma_0} \omega = 0.$$

$(3) \Rightarrow (2)$ Given γ_1, γ_2 , with same end points, let $\tilde{\gamma}_2$ be γ_2 with direction reversed, i.e. $\tilde{\gamma}_2: [-b, -a] \rightarrow U$

$$\tilde{\gamma}_2(t) = \gamma_2(-t).$$

Let γ be concatenation of γ_1 & γ_2 . Then γ is closed. Have

$$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\gamma_1} \omega + \int_{\tilde{\gamma}_2} \omega = \int_{\gamma} \omega \stackrel{(3)}{=} 0,$$

$$\text{so } \int_{\gamma_1} \omega = \int_{\gamma_2} \omega.$$

$(2) \Rightarrow (1)$ Choose $(x_0, y_0) \in U$. For $(x, y) \in U$, let $h(x, y) = \int_{\gamma} \omega$, where γ is a path starting in (x_0, y_0) , ending in (x, y) .
 h is well-defined by assumption (1) .

Claim: $\omega = dh$.

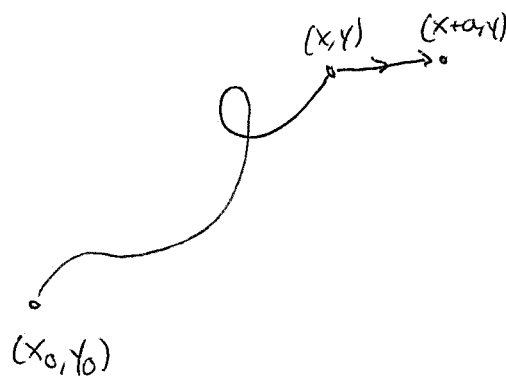
Pf of claim: ~~Assume~~ With $\omega = Pdx + Qdy$, must show $P = \frac{\partial h}{\partial x}$, $Q = \frac{\partial h}{\partial y}$.

Compute $\frac{\partial h}{\partial x}(x, y)$ as follows: Choose path γ from (x_0, y_0) to (x, y) . Let γ_a be straight path ~~from~~ from (x, y) to $(x+a, y)$.

$$\gamma_a(t) = (x+t, y).$$

Then

$$\begin{aligned} h(x+a, y) &= \int_{\gamma} \omega + \int_{\gamma_a} \omega \\ &= h(x, y) + \int_0^a P(x+t, y) dt. \end{aligned}$$



Now

$$\frac{\partial h}{\partial x}(x, y) = \lim_{a \rightarrow 0} \frac{h(x+a, y) - h(x, y)}{a} = \lim_{a \rightarrow 0} \frac{\int_{\gamma_a} \omega}{a} = \lim_{a \rightarrow 0} \frac{\int_0^a P(x+t, y) dt}{a} \\ = P(x, y),$$

as we wanted. $\frac{\partial h}{\partial y} = Q$ is similar.

Closed ~~differentials~~ differentials: Recall, for a differential $\omega = Pdx + Qdy$, □

have

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy.$$

Def A ^{differential} ~~form~~ is closed if $d\omega = 0$.

Thm Every exact differential is closed.

Pf ω exact $\Leftrightarrow$ there is an h s.t. $\omega = dh = \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy$.

Then $d\omega = \left(\frac{\partial}{\partial x} \frac{\partial h}{\partial y} - \frac{\partial}{\partial y} \frac{\partial h}{\partial x} \right) dx dy = 0$, so ω is closed.

Recall a domain U is star-shaped if there is a point $(x_0, y_0) \in U$ s.t. every point (x, y) can be connected to (x_0, y_0) by a straight line segment in U .

Example ~~of~~ Disks, Rectangles, $\mathbb{C}$, half-planes, $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

Non-example: $\mathbb{C} \setminus \{0\}$.

Thm Let U be a star-shaped domain. Then a differential ω is exact if and only if it is closed.

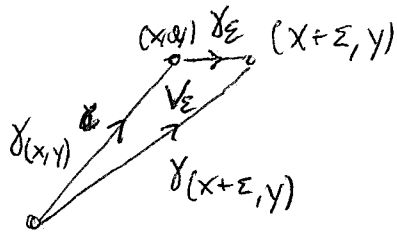
Pf Exact $\Rightarrow$ closed done above.

Closed $\Rightarrow$ exact: Let ω be closed differential, let $(x_0, y_0) \in U$ be "centre" of U . Define $h: U \rightarrow \mathbb{R}$ by $h(x, y) = \int_{\gamma_{(x, y)}} \omega$, where

$$\gamma_{(x, y)}: [0, 1] \rightarrow U \text{ is } \gamma_{(x, y)}(t) = t(x, y) + (1-t)(x_0, y_0).$$

Claim now $dh = \omega$.

Given $(x, y) \in U$ ~~near~~ and $0 < \varepsilon < 1$, can compute $h(x+\varepsilon, y)$ as follows



Green's theorem gives

$$0 = \iint_{V_\varepsilon} d\omega = \iint_{V_\varepsilon} d\omega = \int_{\gamma_{(x+\varepsilon, y)}} \omega - \int_{\gamma_\varepsilon} \omega - \int_{\gamma_{(x, y)}} \omega$$

$$= h(x+\varepsilon, y) - h(x, y) - \int_{\gamma_\varepsilon} \omega,$$

so $h(x+\varepsilon, y) - h(x, y) = \int_{\gamma_\varepsilon} \omega$.

Then find, if $\omega = Pdx + Qdy$, $\frac{\partial h}{\partial x} = P$, similarly $\frac{\partial h}{\partial y} = Q$. $\square$

Example Closed $\nRightarrow$ exact without star-shaped domain.

Let $U = \mathbb{C} \setminus \{0\}$, let $\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$. Then

$$d\omega = \left(+ \frac{\partial}{\partial y} \left(-\frac{y}{x^2+y^2} \right) + \frac{\partial}{\partial x} \left(\frac{x}{x^2+y^2} \right) \right) dx dy$$

$$= \left(\frac{-y^2+x^2}{(x^2+y^2)^2} + \frac{-x^2+y^2}{(x^2+y^2)^2} \right) dx dy = 0,$$

so ω is closed. Let $\gamma: [0, 2\pi] \rightarrow \mathbb{C}$ be $\gamma(t) = (\cos t, \sin t)$. Then

$$\int_\gamma \omega = \int_0^{2\pi} \frac{\sin t}{\cos^2 t + \sin^2 t} (\cos t)' + \frac{\cos t}{\cos^2 t + \sin^2 t} (\sin t)' dt = \int_0^{2\pi} 1 dt = 2\pi \neq 0,$$

So ω is not exact.

Homotopies Let U be a domain, let $\gamma_0, \gamma_1 : [a, b] \rightarrow U$ be paths with $\gamma_0(a) = \gamma_1(a)$, $\gamma_0(b) = \gamma_1(b)$. Say γ_0 and γ_1 are homotopic (in U) if there is a continuous function $F : [0, 1] \times [a, b] \rightarrow U$ such that:

- For every $c \in [0, 1]$, $\gamma_c(t) = F(c, t) : [a, b] \rightarrow U$ is a path with $\gamma_c(a) = \gamma_0(a)$, $\gamma_c(b) = \gamma_0(b)$.
- $F(0, t) = \gamma_0(t)$
- $F(1, t) = \gamma_1(t)$

Thm ~~Let~~ Let U be a domain, ω a closed form on U , and let γ_0, γ_1 be homotopic paths in U . Then

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$$

Proof omitted.