

III.3 - III.5 Harmonic conjugates, the mean value property & the maximum principle.

Let $U \subseteq \mathbb{C}$ be a domain, recall:

- A smooth function $u: U \rightarrow \mathbb{R}$ is harmonic if

$$\frac{\partial^2 u}{(\partial x)^2} + \frac{\partial^2 u}{(\partial y)^2} = 0$$

- A function $v: U \rightarrow \mathbb{R}$ is a harmonic conjugate to u if $f = u + iv$ is analytic. Such a v may or may not exist, if it does it is unique up to addition of a constant.

- A differential $\omega = Pdx + Qdy$ on U is

- closed if $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$

- exact if there is a smooth function h such that

$$dh \stackrel{\text{def}}{=} \frac{\partial h}{\partial x} dx + \frac{\partial h}{\partial y} dy = \omega, \text{ i.e.}$$

$$\frac{\partial h}{\partial x} = P, \quad \frac{\partial h}{\partial y} = Q$$

- ω Exact $\Rightarrow \omega$ closed, always

- If U is star-shaped; ω closed $\Rightarrow \omega$ exact.

Prop Let u be a smooth function on U . Then u is harmonic

if and only if $\omega_u = \frac{\partial u}{\partial y} dx - \frac{\partial u}{\partial x} dy$ is closed.

$$\text{Pf } \omega_u \text{ closed} \Leftrightarrow \frac{\partial}{\partial x} \frac{\partial u}{\partial x} - \frac{\partial}{\partial y} \left(-\frac{\partial u}{\partial y} \right) = 0 \Leftrightarrow \frac{\partial^2}{(\partial x)^2} u + \frac{\partial^2}{(\partial y)^2} u = 0$$

$$\Leftrightarrow u \text{ harmonic.}$$

Prop Let u be harmonic on U , let $\omega_u = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$.

Then v is a harmonic conjugate to u if and only if

$$dv = \omega_u$$

$$\text{Pf } dv \stackrel{\text{def}}{=} \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = \omega_u \stackrel{\text{def}}{=} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$



$$\text{Cauchy-Riemann eqns} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$



v is harmonic conjugate to u . $\square$

Cor A harmonic function u admits a harmonic conjugate if and only if ω_u is exact.

If U is star-shaped, every harmonic function admits a harmonic conjugate.

Pf ω_u exact $\Leftrightarrow \exists v$ s.t. $dv = \omega_u \Leftrightarrow \exists$ harmonic conj. v

U star-shaped + ω_u closed $\Leftrightarrow \omega_u$ exact.

Example Let $U = \mathbb{C} \setminus \{0\}$, $u = \log(x^2 + y^2)$. Then

$$u_x = \frac{2x}{x^2 + y^2} \quad u_y = \frac{2y}{x^2 + y^2}$$

$$u_{xx} = 2 \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad u_{yy} = 2 \frac{x^2 - y^2}{(x^2 + y^2)^2} \Rightarrow u_{xx} + u_{yy} = 0 \Rightarrow u \text{ harmonic}$$

$\omega_u = -\frac{2y}{x^2 + y^2} dx + \frac{2x}{x^2 + y^2} dy$ exact, but not closed, since

$$\oint_{|z|=1} \omega_u = 4\pi \neq 0.$$

u has no harmonic conjugate on U . On any star-shaped open

subset $V \subseteq U$, u has a harmonic conjugate, e.g.

$v = 2 \operatorname{Arg}(x + iy)$ on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, since $u + iv = 2 \operatorname{Log}(z)$.

The mean value property

What are harmonic functions "like"?

Consider harmonic function u , so $u_{xx} + u_{yy} = 0$, say near $(0,0) \in \mathbb{R}^2$. Then $u_{xx}(0,0)$ ~~is related~~ measures of $u(x,0)$ is convex/concave near $x=0$

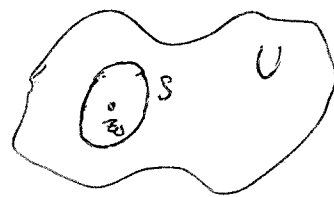
$u_{yy}(0,0)$ measures if $u(0,y)$ is convex/concave near $y=0$

u Harmonic gives $u_{xx} = -u_{yy}$, so convexity of $u(x,0)$ matches concavity of $u(0,y)$, roughly speaking, u cannot curve upwards/downwards in all directions

Two precise mathematical statements of this intuition

Def Let U be a domain, $z_0 \in U$, u a function on U , and $S = \{z \in \mathbb{C} \mid |z - z_0| = r\} \subseteq U$ the circle of radius r centred at z_0 . Then the average value of u on S is

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta.$$



Ex $z_0 = 0$, $r = 1$,

~~Let~~ $u(z) = c$ constant, avg. value on $S = \frac{1}{2\pi} \int_0^{2\pi} c d\theta = c = u(0)$

$$u(z) = x, \text{ avg. } = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{Re} e^{i\theta} d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos \theta d\theta = 0 = u(0)$$

$$u(z) = x^2 + y^2, \text{ avg. } = \frac{1}{2\pi} \int_0^{2\pi} (e^{i\theta})^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} 1 d\theta = 1 \neq u(0)$$

$$u(z) = x^2 - y^2, \text{ avg. } = \frac{1}{2\pi} \int_0^{2\pi} (\cos^2 \theta - \sin^2 \theta) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \cos 2\theta d\theta = 0 = u(0).$$

Thm Let U be a domain, u harmonic on U , $z_0 \in U$, $r_0 > 0$ such that disk $D_{r_0} = \{z \mid |z - z_0| \leq r_0\} \subseteq U$. Then the average value of u on $S_{r_0} = \{z \mid |z - z_0| = r_0\}$ equals $u(z_0)$.

Proof Let $A(r) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$, for $0 \leq r \leq r_0$.

Have $A(0) = u(z_0)$, want $A(0) = A(r_0)$.

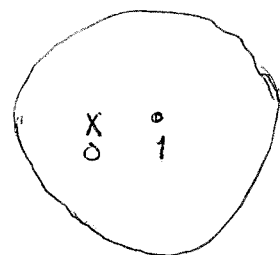
$$\begin{aligned}
\frac{d}{dr} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta &= \int_0^{2\pi} \frac{d}{dr} u(z_0 + re^{i\theta}) d\theta \\
&= \int_0^{2\pi} \cos \theta \frac{\partial u}{\partial x} + \sin \theta \frac{\partial u}{\partial y} d\theta \\
&= \frac{1}{r} \int_0^{2\pi} (y'(t) \frac{\partial u}{\partial x} - x'(t) \frac{\partial u}{\partial y})(x(t), y(t)) dt \\
&= \frac{1}{r} \int_{\gamma} -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy = \frac{1}{r} \int_{\gamma} \omega_u = 0,
\end{aligned}$$

$\gamma(t) = z_0 + re^{it}$ since u is harmonic
 $= (x_0 + r \cos t, y_0 + r \sin t), \quad 0 \leq t \leq 2\pi.$

Thus $A'(r) = 0 \Rightarrow A(r)$ is constant. Hence $A(0) = A(r_0)$. $\square$

Caution Important that u is harmonic on whole disk, e.g.

$u = \frac{-x}{x^2 + y^2}, \quad u(1,0)$ is not equal average over circle
 $\{z \mid |z-1|=2\}$



Maximum principle

Let u be ~~harmonic~~ smooth function on domain U .

To find local maximum of u , procedure is

- Find critical point (x_0, y_0) with $u_x(x_0, y_0) = u_y(x_0, y_0) = 0$
- Compute $\det J_u(x_0, y_0) = \det \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix}(x_0, y_0)$, then

$\det J_u(x_0, y_0) > 0 \begin{cases} \xrightarrow{u_{xx}(x_0, y_0) > 0} (x_0, y_0) \text{ local minimum} \\ \xrightarrow{u_{xx}(x_0, y_0) < 0} (x_0, y_0) \text{ local maximum} \end{cases}$

$\det J_u(x_0, y_0) < 0 \Rightarrow (x_0, y_0)$ saddle point, neither max nor min

$\det J_u(x_0, y_0) = 0 \leadsto$ further analysis needed

If u is harmonic, then $u_{xx} = -u_{yy}$

$$\det \nabla u = \det \begin{pmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{pmatrix} = -(u_{xx}^2 + u_{yy}^2) \leq 0,$$

to 2nd order u is ~~is linear~~ ^{(x_0, y_0) is} a saddle point or u is flat near (x_0, y_0) .

Thm Let u be a ~~harmonic~~ harmonic function on a domain U , and assume u obtains its maximal value at $(x_0, y_0) \in U$. Then u is constant.

Pf Let $M = u(x_0, y_0)$, the assumption is $u(x, y) \leq M$ for all $(x, y) \in U$.

Since U is open, there is an $r > 0$ such that a disk of radius r centred in $z_0 = (x_0, y_0)$ lies in U . Mean value theorem gives

$$\frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta = u(z_0) = M, \text{ also have } u \leq M, \text{ so}$$

$$\int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta \leq \int_0^{2\pi} M d\theta$$

$$v = u - M \leq 0. \text{ Then}$$

$$\frac{1}{2\pi} \int_0^{2\pi} v(z_0 + re^{i\theta}) d\theta = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) - M d\theta = 0, \text{ which means}$$

$$v(z_0 + re^{i\theta}) = 0 \text{ for all } \theta, \text{ hence } u(z_0 + re^{i\theta}) = M \text{ for all } \theta.$$

Similarly $u(z_0 + r'e^{i\theta}) = M$ for all θ , $r' < r$, so ^{on} a whole disk u takes value M .

$$\{z \mid |z - z_0| < r\}$$

U_1

This shows: The set $\{z \in U \mid u(z) = M\}$ is open.

Since u is continuous, $U_2 = \{z \in U \mid u(z) < M\}$ is open.

Now $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$, and U_1, U_2 open, U connected,

gives either $U_1 = \emptyset$ or $U_2 = \emptyset$. $U_1 \neq \emptyset$ by assumption, so

$U_2 = \emptyset$, hence $U_1 = U$, and u is constant.

□

Maximum principle, analytic version Let $f(z)$ be ~~as above~~ analytic on a domain U . If $|f(z)|$ obtains its maximal value, then f is constant.

Note $|f(z)|$ is not harmonic!

Pf Say $|f(z_0)| = M$, $|f(z)| \leq M$ for all $z \in U$.

Then $f(z_0) = M e^{i\theta_0}$, $\theta_0 \in \mathbb{R}$, and $u(z) = \operatorname{Re} \frac{f(z)}{e^{i\theta_0}}$ is

- harmonic

- $u(z_0) = M$

- $u(z) = \operatorname{Re} \frac{f(z)}{e^{i\theta_0}} \leq \left| \frac{f(z)}{e^{i\theta_0}} \right| = |f(z)| \leq M$.

Hence u is constant, so $f(z)/e^{i\theta_0}$ is constant, so $f(z)$ is.

Maximum principle with boundary Let U be a bounded domain, let $f(z)$ be a function analytic on U , continuous on $\bar{U}$. Then $|f(z)|$ obtains its maximal value on the boundary

$$\partial U = \bar{U} \setminus U.$$

Pf If $\bar{U}$ closed and bounded $\Rightarrow |f(z)|$ obtains its maximum.

If $f(z)$ is constant, may choose any point, in particular on ∂U .

Otherwise, $|f(z)|$ does not obtain maximum value in U , hence max. point must be in ∂U .