

### III. 6 - Application to fluid dynamics

Consider a domain  $D \subseteq \mathbb{C}$ , and a fluid flowing in  $D$ .

Examples • Air moving past an airplane wing seen from the side,  
 $D = \mathbb{C} \setminus W$ , where  $W$  is a 2d slice of the wing



• Fluid flowing, seen from above

Represent this by a velocity function  $\vec{V}(z) = (P(z), Q(z))$ , vector field representing velocity of fluid particle at each point.

Assume ①  $\vec{V}$  is independent of time

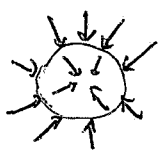
② No source or sink in  $D$

③ Flow is incompressible: Density of fluid is constant ~~and~~ in plane and does not change as time varies

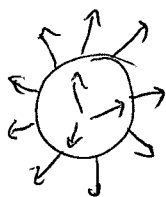
④ Flow is irrotational: No net flow around small loops in  $D$

Intuitively: ② + ③ means ~~no~~ net flow into and out of region in  $D$

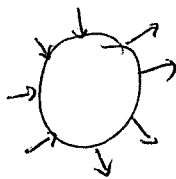
is 0:



No



No



Yes

④ means



No

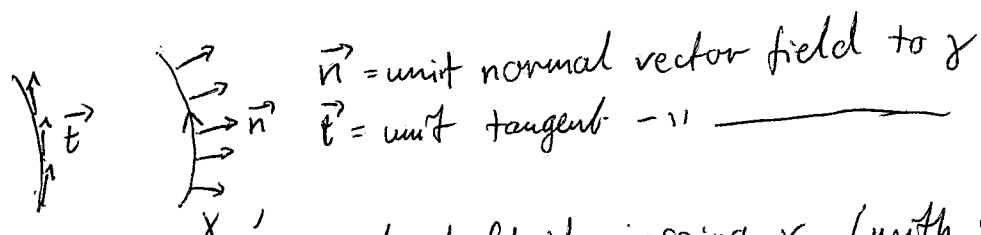


This is ok

Flux and circulation Let  $\gamma: [a, b] \rightarrow D$  be a path, Then the satisfying  $|\gamma'(s)|=1$

• flux across  $\gamma$  of  $V(z)$  is

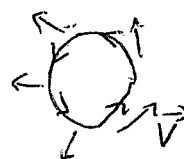
$$\int_a^b \vec{\nabla} \cdot \vec{n} ds = \int_{\gamma} Pdy - Qdx$$



proportional to amount of fluid crossing  $\gamma$  (with sign)

• if  $\gamma$  is closed, the circulation around  $\gamma$  is

$$\int_a^b \vec{\nabla} \cdot \vec{t} ds = \int_{\gamma} Pdx + Qdy.$$



For any small disk  $D_0 \subseteq D$ , with boundary traced by  $\gamma$ ,

$$\textcircled{3} \Rightarrow \text{no flux across } \gamma \Leftrightarrow \int_{\gamma} Pdy - Qdx = 0$$

$$\Leftrightarrow \iint_{D_0} \left( \frac{\partial Q}{\partial y} + \frac{\partial P}{\partial x} \right) dxdy = 0.$$



This holds for any small disk

$$\Rightarrow \frac{\partial Q}{\partial y} + \frac{\partial P}{\partial x} = 0.$$

$$\textcircled{4} \Rightarrow \text{no circulation around } \gamma \Leftrightarrow \int_{\gamma} Pdx + Qdy = 0 \Leftrightarrow \iint_{D_0} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = 0$$

$$\Rightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0.$$

In other words,  $Pdy - Qdx$  &  $Pdx + Qdy$  are closed. On star-shaped subsets (e.g. small disks) of  $D$ , may find functions  $\phi, \psi$  s.t.

$$d\phi = Pdx + Qdy \quad d\psi = Pdy - Qdx$$

Then

$$P = \frac{\partial \phi}{\partial x}, \quad Q = \frac{\partial \phi}{\partial y}$$

$$P = \frac{\partial \psi}{\partial y}, \quad Q = -\frac{\partial \psi}{\partial x}$$

This means  $\phi, \psi$  satisfy ~~ham~~ Cauchy-Riemann equations, so

•  $\phi, \psi$  harmonic

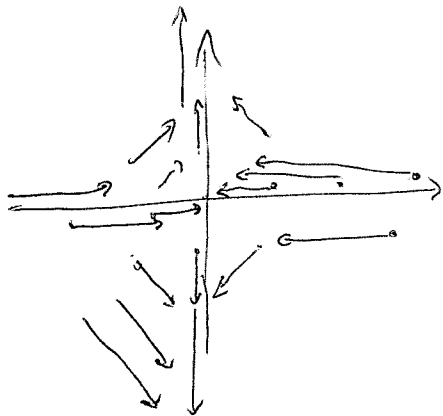
•  $f(z) = \phi + i\psi$  is analytic.

Punchline: ~~On~~ <sup>near any point</sup> At least ~~on subsets~~ of  $D$ , there exists an analytic function  $f(z) = \phi(z) + i\psi(z)$ , called the complex velocity potential, such that

$$V(z) = (P, Q) = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)$$

$$V(z) = P + iQ = \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial x} - i \frac{\partial \psi}{\partial x} = \overline{f'(z)}.$$

Examples  $f(z) = z^2$ ,  $V(z) = -2\bar{z} \leftrightarrow (-2x, 2y)$ ,  $\phi = x^2 - y^2$   
 $\psi = 2xy$



Flux & flow lines Given  $f(z) = \phi + i\psi$ ,  $\vec{V}(z) = \overline{f'(z)}$ , can find

flux across a path ~~& as~~  $\gamma: [a, b] \rightarrow D$

$$\int_{\gamma} P dy - Q dx = \int_{\gamma} d\psi = \psi(b) - \psi(a).$$

Ex Flux ~~over~~ across  $\gamma(t) = 1 + it$   $t \in [-1, 1]$  in example above

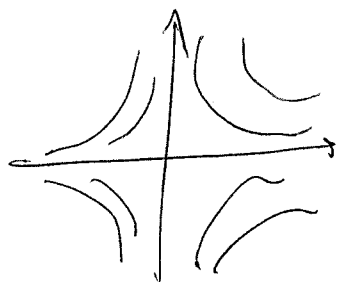
is  $\psi(1+i) - \psi(1-i) = 2 \cdot 1 \cdot 1 - 2 \cdot 1 \cdot (-1) = 4.$

— Flow lines of  $\vec{V}$  are level sets of  $\psi$ :  $\vec{V} = (P, Q) = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right)$ ,

so change in  $\psi$  under moving along a flow line of  $\vec{V}$  is

$$\frac{\partial \phi}{\partial x} \cdot \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \psi}{\partial y} \underset{\substack{\uparrow \\ \text{CR eqns}}}{=} \frac{\partial \phi}{\partial x} \cdot \left( -\frac{\partial \phi}{\partial y} \right) + \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial x} = 0.$$

Ex Flow lines in above example are  $\psi(x, y) = 2xy = c$



Application: Study flows on complicated domains by  $D$  by ~~solve~~ finding simpler domain  $D'$ , along with analytic map  $g: D \rightarrow D'$  to transform solutions from  $D'$  to  $D$ , (e.g. Joukowski transform)