

IV.4–IV.5: Cauchy's theorem and the Cauchy integral formula

1 Primitives

Recall:

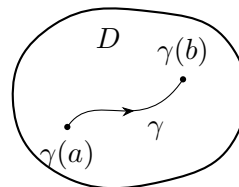
- If $f(z) = u + iv$ is a complex function, then

$$f dz = (u + iv)(dx + i dy) = (u + iv) dx + (-v + iu) dy.$$

- If f is defined on a domain D , then a function F defined on D is a *primitive* for f if
 - F is analytic, and
 - $F'(z) = f(z)$.

Then, for any path $\gamma: [a, b] \rightarrow D$,

$$\int_{\gamma} f(z) dz = F(\gamma(b)) - F(\gamma(a)).$$



Questions.

- When does a function f admit a primitive?
- How unique is such a primitive, if it exists?

Proposition. Let D be a domain, and let f be a continuously differentiable function on D . Then

$$f \text{ is analytic} \iff f dz \text{ is closed.}$$

Proof. Write $f = u + iv$. Then

$$\begin{aligned} f dz \text{ closed} &\iff (u + iv) dx + (-v + iu) dy \text{ closed} \\ &\iff \frac{\partial}{\partial y}(u + iv) = \frac{\partial}{\partial x}(-v + iu) \\ &\iff u_x = v_y \text{ and } u_y = -v_x \quad (\text{Cauchy–Riemann}) \\ &\iff f \text{ analytic.} \end{aligned}$$

□

Proposition. Let D be a star-shaped domain, and $f(z)$ an analytic function on D . Then there is an analytic primitive $F(z)$ for f on D .

Proof. f analytic $\Rightarrow f dz$ closed (by the previous proposition). Since D is star-shaped, closed differentials on D are exact, so there is an F such that $dF = f dz$. We need to show that F is a primitive, i.e.

- F is analytic, and
- $F' = f$.

Write $F = U + iV$ and $f = u + iv$. Then

$$\begin{aligned} dF = f dz &\iff (U_x + iV_x) dx + (U_y + iV_y) dy = (u + iv) dx + (-v + iu) dy \\ &\iff U_x + iV_x = u + iv \text{ and } U_y + iV_y = -v + iu \\ &\iff U_x = V_y = u \text{ and } U_y = -V_x = -v, \end{aligned}$$

which means that the CR equations hold for F , and $F' = U_x + iV_x = u + iv = f$. $\square$

Proposition. If D is a domain and F_1, F_2 are primitives for f , then $F_1 - F_2$ is constant.

Proof. $F'_1 = F'_2 = f \Rightarrow (F_1 - F_2)' = F'_1 - F'_2 = 0 \Rightarrow F_1 - F_2$ is constant. $\square$

Proposition. If F is a primitive for f on a domain D , then for any point $z_0 \in D$ we have

$$F(z) = \int_{z_0}^z f(w) dw + F(z_0),$$

where $\int_{z_0}^z$ denotes the path integral along any path from z_0 to z .

Proof. $\int_{z_0}^z f(w) dw = F(z) - F(z_0)$. $\square$

2 Cauchy's theorem

Theorem (Cauchy's theorem). Let D be a bounded domain with piecewise smooth boundary, and let $f(z)$ be an analytic function on D which extends smoothly to $\overline{D}$. Then

$$\int_{\partial D} f(z) dz = 0,$$

where the integral is taken along a path with D on the left.

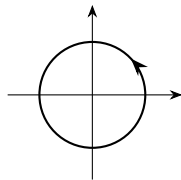
Proof. Green's theorem says

$$\int_{\partial D} f(z) dz = \iint_D d(f(z) dz) = \iint_D 0 = 0,$$

since $f dz$ is closed (Section 1). $\square$

Example. $\oint_{|z|=1} z dz = 0$, since the function z is analytic on

$$D = \{z \in \mathbb{C} \mid |z| < 1\}, \quad \partial D = \{z \in \mathbb{C} \mid |z| = 1\}.$$



Example. Recall that

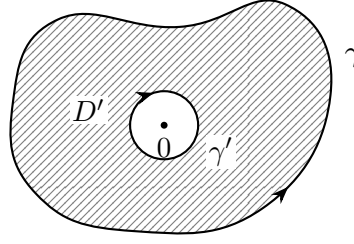
$$\int_{|z|=R} \frac{1}{z} dz = 2\pi i \quad \text{for any } R > 0.$$

This is no contradiction of Cauchy's theorem, since $\frac{1}{z}$ is not analytic at 0.

Consider a curve γ looping once around 0. We claim that

$$\int_{\gamma} \frac{1}{z} dz = 2\pi i.$$

Let D be the domain enclosed by γ , and let $D' = D \setminus V$, where V is a small disk centred at 0.



Then $\frac{1}{z}$ is analytic on D' , so by Cauchy's theorem $\int_{\partial D'} \frac{1}{z} dz = 0$. Now $\partial D' = \gamma \cup \gamma'$, where γ' is a small *clockwise* circle path around 0. This gives

$$0 = \int_{\partial D'} \frac{1}{z} dz = \int_{\gamma} \frac{1}{z} dz + \int_{\gamma'} \frac{1}{z} dz = \int_{\gamma} \frac{1}{z} dz - 2\pi i, \quad \text{so} \quad \int_{\gamma} \frac{1}{z} dz = 2\pi i.$$

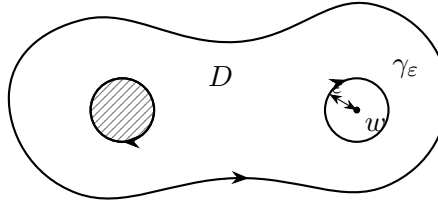
3 The Cauchy integral formula

Theorem (Cauchy integral formula). *Let D be a bounded domain with piecewise smooth boundary, and let $f(z)$ be an analytic function on D extending smoothly to $\overline{D}$. Then for all $w \in D$,*

$$f(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - w} dz.$$

Proof. Let $1 \gg \varepsilon > 0$, and let $B_\varepsilon(w) = \{z \in \mathbb{C} \mid |z - w| \leq \varepsilon\}$. For ε sufficiently small, $B_\varepsilon(w) \subseteq D$. Let $D' = D \setminus B_\varepsilon(w)$. Then

- $\frac{f(z)}{z - w}$ is analytic on D' ;
- $\partial D' = \partial D + \gamma_\varepsilon$, where γ_ε goes around $\partial B_\varepsilon(w) = \{z \mid |z - w| = \varepsilon\}$ clockwise.



Cauchy's theorem gives

$$0 = \int_{\partial D'} \frac{f(z)}{z - w} dz = \int_{\partial D} \frac{f(z)}{z - w} dz + \int_{\gamma_\varepsilon} \frac{f(z)}{z - w} dz.$$

As $\varepsilon \rightarrow 0$, $f(z) \rightarrow f(w)$ on γ_ε , so (details skipped)

$$\int_{\gamma_\varepsilon} \frac{f(z)}{z - w} dz \rightarrow \int_{\gamma_\varepsilon} \frac{f(w)}{z - w} dz = -2\pi i f(w).$$

Thus

$$0 = \int_{\partial D} \frac{f(z)}{z - w} dz - 2\pi i f(w) \implies \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - w} dz = f(w). \quad \square$$

Proposition. *Let D be a bounded domain with piecewise smooth boundary, and let $f(z)$ be analytic on D , extending smoothly to $\overline{D}$. Then*

- f has complex derivatives of all orders, and

- for all $w \in D$,

$$f^{(m)}(w) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^{m+1}} dz.$$

Proof. By induction on m ; the case $m = 0$ is the Cauchy integral formula. Assume $f^{(m)}$ exists and

$$f^{(m)}(w) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^{m+1}} dz.$$

Then

$$\begin{aligned} \lim_{\Delta w \rightarrow 0} \frac{f^{(m)}(w + \Delta w) - f^{(m)}(w)}{\Delta w} &= \frac{m!}{2\pi i} \lim_{\Delta w \rightarrow 0} \int_{\partial D} \frac{\frac{f(z)}{(z-w-\Delta w)^{m+1}} - \frac{f(z)}{(z-w)^{m+1}}}{\Delta w} dz \\ &\stackrel{(*)}{=} \frac{m!}{2\pi i} \int_{\partial D} \lim_{\Delta w \rightarrow 0} \frac{\frac{f(z)}{(z-w-\Delta w)^{m+1}} - \frac{f(z)}{(z-w)^{m+1}}}{\Delta w} dz \\ &= \frac{m!}{2\pi i} \int_{\partial D} \frac{d}{dw} \frac{f(z)}{(z-w)^{m+1}} dz = \frac{m!}{2\pi i} \int_{\partial D} \frac{(m+1)f(z)}{(z-w)^{m+2}} dz \\ &= \frac{(m+1)!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^{m+2}} dz, \end{aligned}$$

showing both that $f^{(m+1)}(w)$ exists and that the formula holds.

(*): ∂D is bounded, and all functions involved are bounded above in size, so the limit can be taken inside the integral. $\square$

Corollary. If $f(z)$ is analytic on some domain D , then f is infinitely complex differentiable, i.e. $f', f'', f''', \dots, f^{(n)}, \dots$ exist.

Proof. To prove the claim at $z_0 \in D$, let D' be a small disk containing z_0 whose closure lies in D , and apply the previous proposition to D' . $\square$

Remark.

- This is surprising, and different from real differentiable functions. For example,

$$f(x) = \begin{cases} 0, & x < 0, \\ x^2, & x \geq 0, \end{cases} \quad \text{has} \quad f'(x) = \begin{cases} 0, & x < 0, \\ 2x, & x \geq 0, \end{cases}$$

but $f''(0)$ does not exist.

- It is hard to prove directly from the definition of analytic.