

LIOUVILLE'S THEOREM, SERIES AND CONVERGENCE OF FUNCTIONS

Recall For a bounded domain D with piecewise smooth boundary, and a function $f(z)$ analytic on D extending smoothly to the boundary ∂D , we have **Cauchy's integral formula**, for $w \in D$

$$f(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-w} dz.$$

Moreover, f admits complex derivatives to all orders, and for the m -th derivative $f^{(m)}$ we have the formula

$$f^{(m)}(w) = \frac{m!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^{m+1}} dz.$$

Let now $w \in \mathbb{C}$, $D = \{z \in \mathbb{C} \mid |z-w| < r\}$, assume f is analytic on D and smooth on ∂D , and let $M \geq 0$ be such that for each $z \in \partial D = \{z \in \mathbb{C} \mid |z-w| = r\}$, we have $|f(z)| \leq M$. Then, for $z \in \partial D$,

$$\left| \frac{f(z)}{(z-w)^{m+1}} \right| \leq \frac{M}{r^{m+1}},$$

and as the length of ∂D is $L = 2\pi r$, the ML estimate gives the **Cauchy estimates** for $f^{(m)}$:

$$|f^{(m)}(w)| = \left| \frac{m!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^{m+1}} dz \right| \leq \frac{m!}{2\pi} \frac{M}{r^{m+1}} 2\pi r = \frac{m!M}{r^m}.$$

Example. Consider $f(z) = z^m$ and $w = 0$. Then $f^{(m)}(z) = m!$. For $|z| = r$, we have that $|f(z)| = |z|^m = r^m$, so we may take $M = r^m$. We then get

$$m! = |f^{(m)}(0)| \leq \frac{m!r^m}{r^m} = m!$$

In particular, we see that the Cauchy estimate is *sharp*, that is, the constant $m!$ cannot be replaced by any smaller one.

Example. Let $f(z) = e^z$, $w = 0$, $r > 0$. Then $f^{(m)}(z) = e^z$. If $|z| = r$, then

$$|e^z| = e^{\operatorname{Re}(z)} \leq e^r,$$

so we may take $M = e^r$ above. The Cauchy estimate becomes the not quite obvious¹ claim

$$1 = |f^{(m)}(0)| \leq \frac{e^r m!}{r^m}.$$

Definition. A function $f(z)$ which is analytic on the domain $\mathbb{C}$ is called **entire**.

Theorem (Liouville's theorem). *If f is a function which is bounded and entire, then f is constant.*

¹You can try to prove this using the power series expansion for e^r .

Proof. The function f being bounded means there is an $M \geq 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let now $w \in \mathbb{C}$ and $r > 0$ be arbitrary. The Cauchy estimate for f' gives

$$|f'(w)| \leq \frac{M}{r}.$$

Since this holds for *all* $r > 0$, we must have $|f'(w)| = 0$. Since $f'(w) = 0$ for all $w \in \mathbb{C}$, we have that f is constant. $\square$

We can now prove the hard part of the fundamental theorem of algebra.²

Theorem. *Every non-constant complex polynomial has a complex zero.*

Proof. Let

$$f(z) = a_n z^n + \cdots + a_0 \quad a_i \in \mathbb{C},$$

with $n \geq 1$ and $a_n \neq 0$. Suppose for a contradiction that $f(z) \neq 0$ for all $z \in \mathbb{C}$. Let $g(z) = \frac{1}{f(z)}$. Then g is entire, and we claim that g is moreover bounded. To see this, note first that, by the triangle inequality

$$|f(z)| = |a_n z^n + \cdots + a_0| \geq |a_n z^n| - |a_{n-1} z^{n-1}| - \cdots - |a_0| = |a_n| |z|^n - \sum_{i=0}^{n-1} |a_i| |z|^i.$$

When $|z| \gg 0$, the term $|a_n| |z|^n$ dominates the other terms in this sum. In particular, there is an $R \geq 0$ such that when $|z| \geq R$, we have

$$|f(z)| \geq |a_n| |z|^n - \sum_{i=0}^{n-1} |a_i| |z|^i \geq 1.$$

Letting $M = \max\{|g(z)| \mid |z| \leq R\}$ (which exists since g is continuous and $\{z \mid |z| \leq R\}$ is closed and bounded), we then have

$$|g(z)| \leq \begin{cases} M & \text{if } |z| \leq R \\ 1/|f(z)| \leq 1 & \text{if } |z| \geq R \end{cases}.$$

Thus g is bounded and entire, hence constant, and hence f is constant, which gives a contradiction. $\square$

SERIES

Spoiler: We'll shortly be using the Cauchy integral formula to show that every analytic function can be locally described as a power series. In order to do this, we need to first recall a few basic facts about convergence and sums of infinite series of complex numbers.

Definition. An infinite series $\sum_{k=0}^{\infty} a_k$ of complex numbers a_k converges to A if the sequence of partial sums $A_n = \sum_{k=0}^n a_k$ converges to A .

Example. For $z \in \mathbb{C}$, we have the infinite sum $\sum_{k=0}^{\infty} z^k$, with partial sums

$$\sum_{k=0}^n z^k = 1 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z},$$

which converges to $1/(1 - z)$ if $|z| < 1$, and does not converge otherwise.

²The easy part being the passage from this to the claim that every complex polynomial factors into linear ones.

Any question about convergence and sums of infinite series translates to a question about convergence and limits of sequences. A few basic observations carry over from sequences:

- If $\sum_{k=0}^{\infty} a_k = A$ and $c \in \mathbb{C}$, then $\sum_{k=0}^{\infty} ca_k = cA$.
- If $\sum_{k=0}^{\infty} a_k = A$ and $\sum_{k=0}^{\infty} b_k = B$, then $\sum_{k=0}^{\infty} (a_k + b_k) = A + B$.

Definition. We say a series $\sum_{k=0}^{\infty} a_k$ converges **absolutely** if $\sum_{k=0}^{\infty} |a_k|$ converges.

Remark. A series $\sum_{k=0}^{\infty} a_k$ converges if and only if the sequence of partial sums A_n is Cauchy, which in turn holds if and only if $A_n - A_m = \sum_{k=m+1}^n a_k$ tends to zero as $m, n \rightarrow \infty$. Since

$$\left| \sum_{k=m+1}^n a_k \right| \leq \sum_{k=m+1}^n |a_k|.$$

If $\sum_{k=0}^{\infty} a_k$ converges absolutely, then $\sum_{k=0}^n |a_k|$ is a Cauchy sequence. Hence the right hand side tends to 0, and so the left hand side also does. In particular, a series which converges absolutely also converges.

Example. The series $\sum_{k=0}^{\infty} z^k$ converges absolutely if $|z| < 1$, since $\sum_{k=0}^{\infty} |z|^k = \sum_{k=0}^{\infty} |z|^k = \frac{1}{1-|z|}$.

Example. The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges to $-\log 2$, but does not converge absolutely, since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

The following tests are useful to prove/disprove convergence of a series:

- **Comparison test** If $|a_k| \leq r_k$ for all $k \geq 0$, and $\sum_{k=0}^{\infty} r_k$ converges, then $\sum_{k=0}^{\infty} |a_k|$ converges, so $\sum_{k=0}^{\infty} a_k$ converges absolutely.
- **Convergence depends on terms going to 0:** If a_k does not tend to 0 as $k \rightarrow \infty$, then $\sum_{k=0}^{\infty} a_k$ does not converge.

Example. The series $\sum_{k=0}^{\infty} z^{k!}$ converges absolutely if $|z| < 1$; we can see this using the comparison test with $|z|^{k!} \leq |z|^k$, since $\sum_{k=0}^{\infty} |z|^k$ converges absolutely.

The series does not converge if $|z| \geq 1$, since then $z^{k!} \not\rightarrow 0$ as $k \rightarrow \infty$.

SEQUENCES OF FUNCTIONS

We now consider sequences and series of functions. Let $E \subseteq \mathbb{C}$, and let $(f_n)_{n=0}^{\infty}$ be a sequence of functions on E . We say

- the sequence (f_n) **converges pointwise** to a function f if, for every $z \in E$, the sequence $(f_n(z))_{n=0}^{\infty}$ converges to $f(z)$.
- the sequence (f_n) **converges uniformly** to a function f if the sequence

$$e_n = \sup\{|f(z) - f_n(z)| \mid z \in E\}$$

converges to 0.³

Remark. We may think of the number e_n above as an “error term” measuring how far away from $f(z)$ the function $f_n(z)$ is at its worst. If f_n converges to f uniformly, then it also converges pointwise, but not vice versa. As a general rule, uniform convergence implies that f inherits nice properties from f_n (e.g. continuity).

³Here we allow $e_n = \infty$.

Example. Take $E = [0, 1] \subset \mathbb{C}$, and let $f_n(x) = x^n$. Defining $f(x)$ by $f(1) = 1$ and $f(x) = 0$ otherwise, we have that for every $x \in [0, 1)$, $f_n(x) = x^n \rightarrow 0$ as $n \rightarrow \infty$, while $f_n(1) = 1 \rightarrow f(1)$ as $n \rightarrow \infty$. On the other hand the error term $e_n = 1$ for all n , so f_n does not converge uniformly to f . Note that f is not continuous even if all f_n are.

Taking partial sums, we get the same notions for *series* of functions, that is, given a series of functions $\sum_{n=0}^{\infty} f_n$, it converges pointwise (resp. uniformly) to a function F if the sequence $F_n = \sum_{k=0}^n f_k$ converges pointwise (resp. uniformly) to F .

Lemma. *If a series $\sum_{n=0}^{\infty} f_n$ of functions on E converges pointwise to F , then the convergence is uniform if and only if the sequence*

$$e_n = \sup \left\{ \left| \sum_{k=n+1}^{\infty} f_k(z) \right| \mid z \in E \right\}$$

converges to 0.

Proof. By definition of uniform convergence, $e_n = \sup\{|F(z) - F_n(z)| \mid z \in E\}$, which equals the right hand side of the equation above. $\square$

Example. On $D = \{z \in \mathbb{C} \mid |z| < 1\}$, the series $\sum_{k=0}^{\infty} z^k$ converges pointwise to the function $\frac{1}{1-z}$. This convergence is not uniform, since for each partial sum $F_n(z) = \sum_{k=0}^n z^k$, the error term

$$\begin{aligned} e_n &= \sup \left\{ \left| \frac{1}{1-z} - F_n(z) \right| \mid z \in D \right\} = \sup \left\{ \left| \sum_{k=n+1}^{\infty} z^k \right| \mid z \in D \right\} \\ &= \sup \left\{ \left| \frac{z^{n+1}}{1-z} \right| \mid z \in D \right\} \end{aligned}$$

is infinite, since $\left| \frac{z^{n+1}}{1-z} \right|$ can be made arbitrarily large by taking $z = 1 - \epsilon$ and ϵ small. On the other hand, replacing D by the closed disc $\overline{D}_r = \{z \in \mathbb{C} \mid |z| \leq r\}$ for some $r < 1$, the series converges uniformly to $\frac{1}{1-z}$, since $\left| \frac{z^{n+1}}{1-z} \right| \leq \frac{r^{n+1}}{1-r} \rightarrow 0$ as $n \rightarrow \infty$.