

CONVERGENCE OF FUNCTIONS, POWER SERIES

We now consider sequences and series of functions. Let $E \subseteq \mathbb{C}$, and let $(f_n)_{n=0}^\infty$ be a sequence of functions on E . We say

- the sequence (f_n) **converges pointwise** to a function f if, for every $z \in E$, the sequence $(f_n(z))_{n=0}^\infty$ converges to $f(z)$.
- the sequence (f_n) **converges uniformly** to a function f if the sequence

$$e_n = \sup\{|f(z) - f_n(z)| \mid z \in E\}$$

converges to 0.¹

Remark. We may think of the number e_n above as an “error term” measuring how far away from $f(z)$ the function $f_n(z)$ is at its worst. If f_n converges to f uniformly, then it also converges pointwise, but not vice versa.

Theorem. Let (f_k) be a sequence of continuous functions on a set E converging uniformly to f . Then

- f is continuous.
- if γ is a path in E , then $\lim_{k \rightarrow \infty} \int_\gamma f_k(z) dz = \int_\gamma f(z) dz$.

Proof. Proof omitted. □

Example. Take $E = [0, 1] \subset \mathbb{C}$, and let $f_n(x) = x^n$. Defining $f(x)$ by $f(1) = 1$ and $f(x) = 0$ otherwise, we have that for every $x \in [0, 1]$, $f_n(x) = x^n \rightarrow f(x)$ as $n \rightarrow \infty$, so $f_n \rightarrow f$ pointwise. On the other hand the error term $e_n = 1$ for all n , so f_n does not converge uniformly to f . Note that f is not continuous even though all f_n are.

Taking partial sums, we get the same notions for *series* of functions: Given a series of functions $\sum_{n=0}^\infty f_n$, it converges pointwise (resp. uniformly) to a function F if the sequence $F_n = \sum_{k=0}^n f_k$ converges pointwise (resp. uniformly) to F .

Remark. From this point on,² let us agree to write

$$B(w, r) = \{z \in \mathbb{C} \mid |z - w| < r\}$$

for the disc of radius r and centre w , so we get the corresponding closed disc

$$\overline{B(w, r)} = \{z \in \mathbb{C} \mid |z - w| \leq r\},$$

and boundary circle

$$\partial B(w, r) = \{z \in \mathbb{C} \mid |z - w| = r\}.$$

Example. On $B(0, 1)$, the series $\sum_{k=0}^\infty z^k$ converges pointwise to the function $\frac{1}{1-z}$. This convergence is not uniform, since for each partial sum $F_n(z) = \sum_{k=0}^n z^k$, the

¹Here we allow $e_n = \infty$.

²This should really have happened sooner in the course; my apologies.

error term

$$\begin{aligned} e_n &= \sup \left\{ \left| \frac{1}{1-z} - F_n(z) \right| \mid z \in B(0,1) \right\} = \sup \left\{ \left| \sum_{k=n+1}^{\infty} z^k \right| \mid z \in B(0,1) \right\} \\ &= \sup \left\{ \left| \frac{z^{n+1}}{1-z} \right| \mid z \in B(0,1) \right\} = +\infty, \end{aligned}$$

since $\left| \frac{z^{n+1}}{1-z} \right|$ can be made arbitrarily large by taking $z = 1 - \epsilon$ and ϵ small. On the other hand, replacing $B(0,1)$ by the closed disc $\overline{B(0,r)}$ for some $r < 1$, the series converges uniformly to $\frac{1}{1-z}$, since $\left| \frac{z^{n+1}}{1-z} \right| \leq \frac{r^{n+1}}{1-r} \rightarrow 0$ as $n \rightarrow \infty$.

Weierstrass M-test.

Theorem. Let $E \subseteq \mathbb{C}$, let $\sum_{k=0}^{\infty} f_k$ be a series of functions on E , and assume that there exist real numbers $M_k \geq 0$ such that $\sum_{k=0}^{\infty} M_k$ converges, and such that $|f_k(z)| \leq M_k$ for all k and $z \in E$. Then $\sum_{k=0}^{\infty} f_k$ converges uniformly.

Proof. For each $z \in E$, $\sum_{k=0}^{\infty} f_k(z)$ converges absolutely to some value $f(z)$, since $|f_k(z)| \leq M_k$ and $\sum_{k=0}^{\infty} M_k$ converges. Using this, we define the function $f(z) = \sum_{k=0}^{\infty} f_k(z)$ on E . For $n \geq 0$, the error term

$$e_n = \sup \left\{ \left| \sum_{k=n+1}^{\infty} f_k(z) \right| \mid z \in E \right\} \leq \sup \left\{ \sum_{k=n+1}^{\infty} |f_k(z)| \mid z \in E \right\} \leq \sum_{k=n+1}^{\infty} M_k.$$

The right hand side tends to 0, so $e_n \rightarrow 0$ and the convergence is uniform. $\square$

Good properties are preserved under uniform convergence.

Theorem. Let $D = B(z_0, r)$ be a disc, let $U \supseteq \overline{D}$ be an open set, and let $(f_k)_{k=0}^{\infty}$ be a sequence of analytic functions on U converging uniformly to f on $\overline{D}$. Then f is analytic on D , and the sequence f'_k converges pointwise to f' on D .

Proof. For each k , we have the Cauchy formula

$$f_k(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f_k(z)}{z-w} dz.$$

We get

$$\begin{aligned} f(w) &= \lim_{k \rightarrow \infty} f_k(w) = \lim_{k \rightarrow \infty} \frac{1}{2\pi i} \int_{\partial D} \frac{f_k(z)}{z-w} dz \\ &= \frac{1}{2\pi i} \int_{\partial D} \lim_{k \rightarrow \infty} \frac{f_k(z)}{z-w} dz = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-w} dz. \end{aligned}$$

Here the third equality is justified since the limit is uniform. Now from the expression

$$f(w) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z-w} dz,$$

differentiating with respect to w , we can check that f is analytic at w with

$$\begin{aligned} f'(w) &= \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{(z-w)^2} dz = \frac{1}{2\pi i} \int_{\partial D} \lim_{k \rightarrow \infty} \frac{f_k(z)}{(z-w)^2} dz \\ &= \lim_{k \rightarrow \infty} \frac{1}{2\pi i} \int_{\partial D} \frac{f_k(z)}{(z-w)^2} dz = \lim_{k \rightarrow \infty} f'_k(w). \end{aligned}$$

$\square$

POWER SERIES

A **power series** is a function in z given by a series of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k \quad z_0, a_k \in \mathbb{C}.$$

Convergence of power series is very well behaved:

Theorem. *For any power series $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ there is an $R \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that*

- *if $|z - z_0| < R$, the series converges absolutely*
- *if $|z - z_0| > R$, the series does not converge*

We call R the **radius of convergence** of the power series. If $|z - z_0| = R$, we cannot say much in general: the series may converge non-absolutely, converge absolutely, or not converge at all.

Proof. Let

$$S = \{r \in \mathbb{R}_{\geq 0} \mid \text{the sequence } |a_k| r^k \text{ is bounded}\}.$$

Clearly, $0 \in S$, and if $r' < r$ with $r \in S$, then $r' \in S$. We let $R = \sup S$, taking $R = +\infty$ if $S = \mathbb{R}_{\geq 0}$. Then if $r < R$, the sequence $|a_k| r^k$ is bounded, and if $r > R$, it is unbounded.

Assume now $|z - z_0| = r < R$, and let $r' \in (r, R)$. Then

$$|a_k (z - z_0)^k| = |a_k| r^k = |a_k| r'^k \left(\frac{r}{r'}\right)^k.$$

Let M be an upper bound for $|a_k| r'^k$; we then have

$$|a_k| r'^k \left(\frac{r}{r'}\right)^k \leq M \left(\frac{r}{r'}\right)^k.$$

Since $r/r' < 1$, the series $\sum M(r/r')^k$ converges, and so the series $\sum a_k (z - z_0)^k$ converges absolutely.

Assume next that $|z - z_0| = r > R$. Then the sequence $|a_k| r^k$ is unbounded. In particular $a_k (z - z_0)^k$ does not converge to 0 as $k \rightarrow \infty$, hence $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ does not converge. $\square$

Remark. We may extract the following exact formula (the “Cauchy–Hadamard formula”) for R from the proof:

$$R = \frac{1}{\limsup_k |a_k|^{1/k}},$$

where $1/0 = \infty$ and $1/\infty = 0$.

Also useful are the following two convergence tests:

- **Ratio test:** If $\lim_{k \rightarrow \infty} |a_k|/|a_{k+1}|$ exists (possibly as ∞), then $R = \lim_{k \rightarrow \infty} |a_k|/|a_{k+1}|$.
- **Root test:** If $\lim_{k \rightarrow \infty} |a_k|^{1/k}$ exists, then $R = \frac{1}{\lim_{k \rightarrow \infty} |a_k|^{1/k}}$ (here again $1/0 = \infty$).

As functions, power series are well behaved:

Theorem. *Let $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ be a power series with radius of convergence R . Then $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ is analytic on $B(z_0, R)$, and we have $f'(z) = \sum_{k=0}^{\infty} (k+1) a_{k+1} (z - z_0)^k$.*

Proof. Let $w \in B(z_0, R)$, and let $r < R$ be such that $w \in B(z_0, r)$. On $\overline{B(z_0, r)}$ the convergence of $\sum_{k=0}^n a_k(z - z_0)^k$ to $f(z)$ as $n \rightarrow \infty$ is uniform, by the Weierstrass M-test applied to $M_k = |a_k|r^k$, and so it follows both that f is analytic at w , and that

$$f'(w) = \lim_{n \rightarrow \infty} \frac{d}{dz} \left(\sum_{k=0}^n a_k(z - z_0)^k \right) \Big|_{z=w} = \sum_{k=0}^{\infty} (k+1)a_{k+1}(w - z_0)^k.$$

□