

POWER SERIES EXPANSIONS

Recall that a **power series** is a series of the form

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k,$$

where $a_k, z_0 \in \mathbb{C}$ are given. A power series has a *radius of convergence* R , such that if $|z - z_0| < R$, the series converges absolutely, while if $|z - z_0| > R$, the series does not converge.

Theorem (“Power series are analytic functions”). *Let $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ be a power series with radius of convergence $R > 0$. Then $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ is analytic on $B(z_0, R)$, and we have $f'(z) = \sum_{k=0}^{\infty} (k+1)a_{k+1}(z - z_0)^k$.*

Proof. Let $w \in B(z_0, R)$, and let $r < R$ be such that $w \in B(z_0, r)$. On $\overline{B(z_0, r)}$ the convergence of $\sum_{k=0}^n a_k (z - z_0)^k$ to $f(z)$ as $n \rightarrow \infty$ is uniform, by the Weierstrass M-test applied to $M_k = |a_k| r^k$, since $\sum_k M_k$ converges as $r < R$. It follows both that f is analytic at w , and that

$$f'(w) = \lim_{n \rightarrow \infty} \frac{d}{dz} \left(\sum_{k=0}^n a_k (z - z_0)^k \right) \Big|_{z=w} = \sum_{k=0}^{\infty} (k+1)a_{k+1}(w - z_0)^k.$$

□

The power series expansion of an analytic function. By the above, power series are analytic functions within the disc of convergence. In fact, a sort of converse to this holds, namely every function analytic at a point $z_0 \in \mathbb{C}$ can be described by a power series near z_0 .

Theorem. *Let $f(z)$ be an analytic function on $B(z_0, r)$.¹ Then, with*

$$a_k = \frac{f^{(k)}(z_0)}{k!},$$

the power series

$$\sum_{k=0}^{\infty} a_k (z - z_0)^k$$

has radius of convergence $R \geq r$, and we have

$$f(w) = \sum_{k=0}^{\infty} a_k (w - z_0)^k$$

for all $w \in B(z_0, r)$.

¹We allow $r = +\infty$, in which case $B(z_0, r) = \mathbb{C}$.

Proof. We assume for simplicity that $z_0 = 0$, let $w \in B(0, r)$, and choose s with $|w| < s < r$. Then the Cauchy integral formula gives

$$(1) \quad f(w) = \frac{1}{2\pi i} \int_{\partial B(0, s)} \frac{f(z)}{z - w} dz = \frac{1}{2\pi i} \int_{\partial B(0, s)} \frac{f(z)}{z} \frac{1}{1 - w/z} dz = \frac{1}{2\pi i} \int_{\partial B(0, s)} \sum_{k=0}^{\infty} \frac{f(z)w^k}{z^{k+1}} dz.$$

On $\partial B(0, s)$, the function $\frac{f(z)}{z}$ is continuous, hence bounded, so there is some M such that $\left| \frac{f(z)}{z} \right| \leq M$ for all $z \in \partial B(0, s)$. Then

$$\left| \frac{f(z)w^k}{z^{k+1}} \right| \leq M \left| \frac{w}{z} \right|^k = M \left(\frac{|w|}{s} \right)^k.$$

Since $|w| < s$, we have $\frac{|w|}{s} < 1$, and so

$$\sum_{k=0}^{\infty} M \left(\frac{|w|}{s} \right)^k$$

converges. By the Weierstrass M-test, the series

$$\sum_{k=0}^{\infty} \frac{f(z)w^k}{z^{k+1}}$$

then converges uniformly as a function of z on $\partial B(0, s)$. We can therefore interchange the summation and integral signs, and get

$$(2) \quad \frac{1}{2\pi i} \int_{\partial B(0, s)} \sum_{k=0}^{\infty} \frac{f(z)w^k}{z^{k+1}} dz = \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{\partial B(0, s)} \frac{f(z)w^k}{z^{k+1}} dz.$$

By the Cauchy integral formula for derivatives, we find

$$(3) \quad \frac{1}{2\pi i} \sum_{k=0}^{\infty} \int_{\partial B(0, s)} \frac{f(z)w^k}{z^{k+1}} dz = \sum_{k=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial B(0, s)} \frac{f(z)}{z^{k+1}} dz \right) w^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} w^k.$$

Combining equations (1), (2) and (3) and letting $a_k = \frac{f^{(k)}(0)}{k!}$, we find that

$$(4) \quad f(w) = \sum_{k=0}^{\infty} a_k w^k,$$

which is what we wanted.

Since (4) holds for all w with $|w| < r$, in particular the series converges for all such w . This implies that its radius of convergence R is at least r . $\square$

Example. For $f(z) = e^z$, we have $f^{(k)}(z) = e^z$ and so $f^{(k)}(0) = 1$ for all $k \geq 0$. For the power series expansion of e^z around 0, we therefore get $a_k = \frac{1}{k!}$, and so

$$e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

This power series has convergence radius $+\infty$; that is, it converges for all $z \in \mathbb{C}$.

We can see this either by computing the convergence radius using the ratio test, by

$$\lim_{k \rightarrow \infty} \frac{|a_k|}{|a_{k+1}|} = \lim_{k \rightarrow \infty} \frac{1/k!}{1/(k+1)!} = \lim_{k \rightarrow \infty} (k+1) = +\infty,$$

or directly from the theorem: since e^z is analytic on $\mathbb{C} = B(0, +\infty)$, the series has convergence radius $+\infty$.

The derivatives of f at a point determine f completely near it.

Proposition. *Let f and g be analytic functions on $B(z_0, r)$. If $f^{(k)}(z_0) = g^{(k)}(z_0)$ for all $k \geq 0$, then $f = g$.*

Proof. Letting $a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{g^{(k)}(z_0)}{k!}$, we have

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k = g(z)$$

for all $z \in B(z_0, r)$. □

Proposition. *Let f be analytic on $B(z_0, r)$, and let $\sum_{k=0}^{\infty} a_k (z - z_0)^k$ be its power series expansion at z_0 . Then the radius of convergence R is the maximum possible value of s for which f admits an extension to an analytic function on $B(z_0, s)$.*

Proof. The power series itself defines an analytic extension of f on $B(z_0, R)$. Conversely, if g is an analytic extension of f to $B(z_0, s)$, then $g(z) = f(z)$ for $z \in B(z_0, r)$, so $g^{(k)}(z_0) = f^{(k)}(z_0)$, hence g and f have the same power series expansion at z_0 . Since g is analytic on $B(z_0, s)$, the power series expansion theorem applied to g gives $R \geq s$. □

It is in general somewhat subtle to understand whether a function f can be extended analytically to a given disc, as the following examples show.

Example. Consider the function $f(z) = \frac{1}{1+z^2}$, with its power series expansion at 0. Here we can set $w = -z^2$ and use the geometric series

$$f(z) = \frac{1}{1-w} = \sum_{k=0}^{\infty} w^k = \sum_{k=0}^{\infty} (-z^2)^k = 1 - z^2 + z^4 - z^6 + \dots,$$

which converges whenever $|z| < 1$ and diverges when $|z| > 1$.

The radius of convergence being 1 could have been predicted as follows: $f(z)$ is analytic on $B(0, 1)$, and so its power series has radius of convergence at least 1. On the other hand, as z approaches i along the imaginary axis, we have

$$f(i - i\epsilon) = \frac{1}{1 + (i - i\epsilon)^2} = \frac{1}{2\epsilon - \epsilon^2} \rightarrow \infty$$

as $\epsilon \rightarrow 0$. Hence there is no way to extend f analytically (or even continuously) to a disc centred at 0 of radius > 1 .

Example. Let $f(z) = \frac{\sin(z)}{z}$, which is defined for all $z \in \mathbb{C} \setminus \{0\}$. For any $z_0 \in \mathbb{C} \setminus \{0\}$, f is analytic on $B(z_0, |z_0|)$, and so the series expansion of f about $z_0 \in \mathbb{C} \setminus \{0\}$ has convergence radius at least $|z_0|$.

However, we may in fact extend f to be analytic also at 0, as follows. For all $z \in \mathbb{C}$, we have that

$$\sin(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1},$$

and so, for all $z \neq 0$ that

$$f(z) = \frac{\sin(z)}{z} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} z^{2k}.$$

The power series on the right hand side converges for all z , and so defines an extension of f to the entire complex plane.

Example. The function $e^{-1/z}$ is analytic on $B(1, 1)$, but cannot be extended to a continuous function on a disc of radius $r > 1$ centred at 1.² It is, however, bounded, since $|e^{-1/z}| = e^{-\operatorname{Re}(1/z)} < e^{-1/2}$ for $z \in B(1, 1)$; this shows that the “blowing up to ∞ ” phenomenon from the $1/(1+z^2)$ example is not the only obstruction to extending a function in an analytic way.

The convergence radius of the power series expansion of $e^{-1/z}$ about $z_0 = 1$ is 1.

The three examples above will return as examples of the three kinds of “singularities” an analytic function can have; respectively, they are a pole ($\frac{1}{1+z^2}$ at i), a removable singularity ($\sin(z)/z$ at 0) and an essential singularity ($e^{-1/z}$ at 0).

²If you like, show this by computing $\operatorname{Re} e^{-1/z}$ for $z \in \partial B(1, 1) \setminus \{0\}$ and $\lim e^{-1/x}$ as $x \rightarrow 0^+$.