ISOLATED SINGULARITIES OF ANALYTIC FUNCTIONS

We say a property of a function holds **near a point** $z_0 \in \mathbb{C}$ if there is some $r > 0$ such that the property holds on

$$B(z_0, r) \setminus \{z_0\} = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\}$$

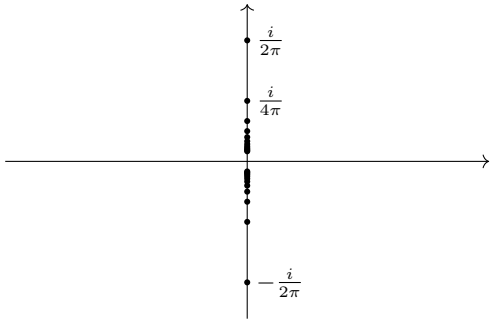
Definition. Let $f(z)$ be an analytic function on some domain. We say f has an **isolated singularity** at z_0 if it is analytic near z_0, but $f(z_0)$ is not defined. To be precise, this means there is an $r > 0$ such that f is analytic on

$$B(z_0, r) \setminus \{z_0\} = \{z \in \mathbb{C} \mid 0 < |z - z_0| < r\},$$

Examples.

- $f(z) = \dfrac{1}{z}$ is analytic near 0 (e.g. on $B(0,1) \setminus \{0\}$), but not defined in 0, so has an isolated singularity at 0.
- $\dfrac{1}{e^z - 1}$ has isolated singularities at $\{2\pi i k\}_{k \in \mathbb{Z}}$.
- $f(z) = \dfrac{1}{e^{1/z} - 1}$ is analytic where $e^{1/z} \neq 1$, i.e. where $1/z \notin 2\pi i\mathbb{Z}$, i.e. where

$$z \neq \frac{1}{2\pi i k} \quad \text{for } k \in \mathbb{Z} \setminus \{0\}.$$



This gives an isolated singularity at each point of $\left\{\frac{1}{2\pi i k} \;\middle|\; k \in \mathbb{Z} \setminus \{0\}\right\}$. Note that $f(0)$ is not defined, but f is not analytic near 0, so f does not have an isolated singularity at 0.

Laurent series. If f has an isolated singularity at z_0, then there is a Laurent expansion of f near z_0,

$$f(z) = \sum_{k \in \mathbb{Z}} a_k (z - z_0)^k,$$

valid for all z with $0 < |z - z_0| < R$ for some $R > 0$.

Definition. The **principal part** of the Laurent expansion at z_0 is the sum of the negative order terms,

$$\sum_{k=-1}^{-\infty} a_k (z - z_0)^k.$$

There are 3 cases:

- The principal part is 0; $a_k = 0$ for all $k < 0$: say the singularity is *removable*.

- The principal part has finitely many terms; there is an $n > 0$ such that $a_{-n} \neq 0$ and $a_k = 0$ for all $k < -n$,
$$f(z) = a_{-n}(z - z_0)^{-n} + a_{-n+1}(z - z_0)^{-n+1} + \cdots;$$
say the singularity is a **pole of order** n.
- The principal part has infinitely many terms, i.e. there are infinitely many $k < 0$ such that $a_k \neq 0$; say the singularity is **essential**.

Examples.

- $f(z) = \dfrac{\sin z}{z}$ has Laurent expansion at 0 given by
$$\frac{\sin z}{z} = \frac{\displaystyle\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!}}{z} = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k+1)!} = 1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots,$$
so the singularity is removable.
- $f(z) = \dfrac{1}{z} = z^{-1}$ near 0, so it has a pole of order 1 at 0.
- $f(z) = \dfrac{1}{\sin z}$:

$$\frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{6} + \cdots} = \frac{1}{z} \cdot \frac{1}{1 - \frac{z^2}{6} + \cdots} = \frac{1}{z}\left(1 + \left(\frac{z^2}{6} - \cdots\right) + \left(\frac{z^2}{6} - \cdots\right)^2 + \cdots\right)$$
$$= \frac{1}{z} + \frac{z}{6} + \cdots.$$
The principal part of $1/\sin(z)$ at 0 is $1/z$, so the function has a pole of order 1 at 0.
- $f(z) = e^{1/z} = \displaystyle\sum_{k=0}^{\infty} \frac{z^{-k}}{k!}$ for all $z \neq 0$, so it has an essential singularity at 0.

Removable singularities. If f has a removable singularity at z_0, then setting $f(z_0) = a_0$ gives an analytic extension of f to z_0.

Theorem. An isolated singularity of f at z_0 is removable if and only if f is bounded near z_0.

Here "f is bounded near z_0" means there exist $r, M > 0$ such that
$$0 < |z - z_0| < r \implies |f(z)| \leq M.$$

Proof. If the singularity is removable, then f can be extended continuously to z_0, hence is bounded near z_0.[1]

If $|f|$ is bounded by M near z_0, take $0 < r \ll 1$ and compute the coefficients of $f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$ by
$$a_n = \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)(z - z_0)^{-n-1}\, dz,$$
since, by uniform convergence,
$$\oint_{|z-z_0|=r} f(z)(z - z_0)^{-n-1}\, dz = \oint_{|z-z_0|=r} \sum_{k=-\infty}^{\infty} a_k(z - z_0)^{k-n-1}\, dz$$
$$= \sum_{k=-\infty}^{\infty} a_k \oint_{|z-z_0|=r} (z - z_0)^{k-n-1}\, dz = 2\pi i\, a_n,$$

[1]There is a δ such that when $|z - z_0| < \delta$, we have $|f(z) - f(z_0)| < 1$, hence by the triangle inequality $|f(z)| \leq |f(z) - f(z_0)| + |f(z_0)| < |f(z_0)| + 1$ for all $z \in B(z_0, \delta)$.

since

$$\oint_{|z-z_0|=r} (z-z_0)^j\,dz = \begin{cases} 2\pi i & \text{if } j=-1,\\ 0 & \text{if } j\neq -1. \end{cases}$$

For $n<0$ we get

$$|a_n| = \left|\frac{1}{2\pi i}\oint_{|z-z_0|=r} f(z)(z-z_0)^{-n-1}\,dz\right| \le \frac{1}{2\pi}\big(Mr^{-n-1}\big)(2\pi r) = Mr^{-n} \longrightarrow 0$$

as $r\to 0$, using the ML-estimate. Hence $a_n=0$, and the singularity is removable. □

Poles.

Theorem. Let f have an isolated singularity at z_0. Then the following are equivalent:

① f has a pole of order n.

② There is a function g, analytic in z_0, with $g(z_0)\neq 0$, such that near z_0

$$f(z) = (z-z_0)^{-n}g(z).$$

③ There is a function h, analytic in z_0, with a zero of order n in z_0, such that near z_0

$$f(z) = \frac{1}{h(z)}.$$

Proof. 1 ⇒ 2: f has a pole of order n means $f(z) = a_{-n}(z-z_0)^{-n} + a_{-n+1}(z-z_0)^{-n+1} + \cdots$. Take

$$g(z) = \sum_{k=-n}^{\infty} a_k (z-z_0)^{n+k}.$$

2 ⇒ 3: $g(z)$ is analytic at z_0 and $g(z_0)\neq 0$. Let

$$h(z) = (z-z_0)^n \frac{1}{g(z)}.$$

Then h has a zero of order n in z_0 and $f(z) = \frac{1}{h(z)}$ near z_0.

3 ⇒ 1: h is analytic at z_0 and has a zero of order n in z_0, so $h(z) = a_n(z-z_0)^n + \cdots$ near z_0, with $a_n\neq 0$. Thus

$$h(z) = (z-z_0)^n h_0(z),$$

where h_0 is analytic at z_0 and $h_0(z_0) = a_n \neq 0$. Then $\frac{1}{h_0(z)}$ is analytic at z_0, so

$$\frac{1}{h_0(z)} = b_0 + b_1(z-z_0) + \cdots \qquad \Big(b_0 = \frac{1}{a_n}\Big).$$

We get

$$f(z) = \frac{1}{h(z)} = \frac{1}{(z-z_0)^n h_0(z)} = b_0(z-z_0)^{-n} + b_1(z-z_0)^{-n+1} + \cdots,$$

as wanted. □

Theorem. Let f be an analytic function with an isolated singularity at z_0. Then f has a pole at z_0 if and only if $|f(z)|\to\infty$ as $z\to z_0$.

Proof. If f has a pole, then write $f(z) = (z-z_0)^{-n}g(z)$ as above, with $g(z_0)\neq 0$. Then

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \frac{g(z)}{(z-z_0)^n} = \infty.$$

If $f(z)\to\infty$ as $z\to z_0$, then $\frac{1}{f(z)}\to 0$ as $z\to z_0$. In particular, $\frac{1}{f}$ is bounded and well-defined near z_0, since $f\neq 0$ near z_0, so it extends to an analytic function h

with $h(z_0) = 0$. The function h has a zero of some order n at z_0, and so $f = 1/h$ has a pole of order n at z_0. □

Essential singularities.

Theorem (Casorati–Weierstrass)**.** Assume f has an essential singularity at z_0. Then the following two claims hold

(1) For every $w \in \mathbb{C}$ and $\epsilon_w, \epsilon_z > 0$, there is a $z \in B(z_0, \epsilon_z)$ such that $f(z) \in B(w, \epsilon_w)$.
(2) For every $w \in \mathbb{C}^*$, there is a sequence $(z_i)_{i=1}^{\infty}$ such that $z_i \to z_0$ and $f(z_i) \to w$ as $i \to \infty$.

Example. $f(z) = e^{1/z}$ has an essential singularity at 0. For every $w \in \mathbb{C}^*$ we can find a sequence $z_1, z_2, \ldots \to 0$ such that $e^{1/z_i} \to w$:

$$w = 1, \quad z_k = \frac{1}{k \cdot 2\pi i} \qquad w = 0, \quad z_k = -\frac{1}{k}$$
$$w = 2, \quad z_k = \frac{1}{\log 2 + 2k\pi i} \qquad w = \infty, \quad z_k = \frac{1}{k}$$

Proof. Assume $w \in \mathbb{C}$, and omit $w = \infty$ for simplicity. We first show that (1) $\implies$ (2). By (1), for every integer $i \geq 1$, there is a $z_i \in B(z_0, 1/i)$ such that $f(z_i) \in B(w, 1/i)$. These form a sequence such that $z_i \to z_0$ and $f(z_i) \to w$, proving (2).

We now prove (1). Suppose for a contradiction ϵ_w, ϵ_z are such that the claim fails. Then for all $z \in B(z_0, \epsilon_z) \setminus \{z_0\}$, we have $|f(z) - w| \geq \epsilon_w$. Let now

$$h(z) = \frac{1}{f(z) - w}.$$

Then on $B(z_0, \epsilon_z) \setminus \{z_0\}$ we have

$$|h(z)| = \frac{1}{|f(z) - w|} \leq \frac{1}{\epsilon_w};$$

in particular h is analytic and bounded near z_0. But then h extends over z_0, either with $h(z_0) \neq 0$ or with a zero of order n at z_0. In the first case,

$$f = \frac{1}{h} + w$$

has a removable singularity at z_0, and in the second it has a pole of order n. This is a contradiction. □