MEROMORPHIC FUNCTIONS

Definition. Let $f(z)$ be a function defined on some open set $U \subseteq \mathbb{C}$. We say f has an **isolated singularity at** ∞ if f is analytic on some set

$$\{z \in \mathbb{C} \mid |z| > R\}.$$

If f has an isolated singularity at ∞, then the function $g(w) = f(1/w)$ is analytic on the set $\{w \mid 0 < |w| < R^{-1}\}$, and so g has an isolated singularity at 0. We say the singularity of f at ∞ is removable, a pole, or an essential singularity, respectively, according to what type of singularity g has at 0.

Example. Every entire function f has an isolated singularity at ∞. Writing

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

we get

$$g(w) = \sum_{k=0}^{\infty} a_k w^{-k},$$

so that at the point 0, g has (and at the point ∞, f has)

- A removable singularity if $a_k = 0$ for all $k \geq 1$, that is, if $f(z) = a_0$.
- A pole of order n if there is a maximal $n \geq 1$ such that $a_n \neq 0$; that is, if f is a polynomial of degree n.
- An essential singularity otherwise.

Example. The function $f(z) = 1/(1+z)$ is analytic for $|z| > 1$, so has an isolated singularity at ∞. Then

$$g(w) = f(1/w) = \frac{1}{1+1/w} = \frac{w}{1+w},$$

which has a removable singularity at 0, so f has a removable singularity at ∞.

Definition. A function f on a domain $D \subseteq \mathbb{C}$ is called **meromorphic** if it is analytic except for possible isolated singularities, all of which are poles.

If D contains a set $\{z \mid |z| > R\}$, we say that f is **meromorphic** on $D \cup \{\infty\}$ if it is meromorphic on D and has an isolated singularity at ∞, which is either removable or a pole.

Example. The function $1/(1+z)$ has a pole at -1 and no other singularities, so is meromorphic on $\mathbb{C}$. It has a removable singularity at ∞, so is also meromorphic on $\mathbb{C}^*$.

Example. The function $\frac{1}{\sin z}$ has a pole at each point $k\pi, k \in \mathbb{Z}$, and no other singularities. It is therefore meromorphic on $\mathbb{C}$. It is not analytic on any set $\{z \mid |z| > R\}$, since every such set contains a point $k\pi$, and so it is not meromorphic on $\mathbb{C}^*$.

Theorem. *A function f defined on $\mathbb{C}$ is meromorphic on $\mathbb{C}^*$ if and only if it is rational, that is, if $f(z) = p(z)/q(z)$ where p and q are polynomials.*

Proof. Assume f is a non-zero rational function, so we may write

$$f(z) = \frac{p(z)}{q(z)} = \frac{\sum_{k=0}^{n} a_k z^k}{\sum_{k=0}^{m} b_k z^k},$$

with $a_n \neq 0 \neq b_m$. Then f is analytic on $\mathbb{C}$ except at the zeros of q, where it has poles or removable singularities. In particular, it is analytic when $|z|$ is sufficiently large, and so has an isolated singularity at ∞. To analyse f near ∞, we let $g(w) = f(w^{-1})$, and get

$$g(w) = \frac{p(1/w)}{q(1/w)} = \frac{\sum_{k=0}^{n} a_k w^{-k}}{\sum_{k=0}^{m} b_k w^{-k}} = \left(\frac{a_n}{b_m} w^{m-n}\right) \frac{1 + w(\dots)}{1 + w(\dots)},$$

where $(\dots)$ denotes some polynomial in w. We see that as $w \to 0$, g is bounded (if $m \geq n$) or goes to ∞ (if $m < n$), so g has either a removable singularity or a pole at 0. Hence f has either a removable singularity or a pole at ∞.

Suppose now f is meromorphic on $\mathbb{C}^*$. Then there is an $R \geq 0$ such that f is analytic on $\{z \mid |z| > R\}$. In particular, the singularities of f in $\mathbb{C}$ are all contained in $\overline{B(0,R)}$. This means there are at most finitely many of them, since if there were infinitely many, there would be a point $z_0 \in \overline{B(0,R)}$ which is a limit of a sequence of other singular points of f; this means that f cannot be analytic or have an isolated singularity at z_0. Letting $z_1, \dots, z_n \in \overline{B(0,R)}$ be the poles of f, with the order of the pole at z_i equal to e_i, we let

$$p(z) = \prod_{i=1}^{n} (z - z_i)^{e_i} f(z).$$

Then p has removable singularities at each z_i, and so can be extended to an entire function.

Now as f has either a pole or a removable singularity at ∞, the same is true of p, and so p must be a polynomial, by our previous example. It follows that

$$f(z) = \frac{p(z)}{q(z)}, \qquad q(z) = \prod_{i=1}^{n} (z - z_i)^{e_i},$$

so f is rational. □

Recall that for a function f with an isolated singularity at z_i, we call

$$P_{z_i}(z) = \sum_{k=-\infty}^{-1} a_k (z - z_i)^k$$

its **principal part** at z_i, the sum of the negative order terms of the Laurent series of f at z_i.

Corollary. *Every rational function $f(z)$ admits a partial fraction decomposition; specifically, we can write*

$$f(z) = \sum_{i=1}^{n} P_{z_i}(z) + P(z),$$

where $z_1, \dots, z_n$ denote the poles of f, and P is a polynomial.

Proof. The function $f - \sum_{i=1}^{n} P_{z_i}$ has removable singularities at $z_1, \dots, z_n$, and no other singularities, hence is entire. It is a rational function, hence meromorphic on $\mathbb{C}^*$, and hence is a polynomial. □

Example. The rational function

$$f(z) = \frac{z}{1+z^2} = \frac{z}{(z+i)(z-i)}$$

has simple poles at $i, -i$, with principal parts

$$P_i(z) = \frac{1}{2}(z-i)^{-1}, P_{-i}(z) = \frac{1}{2}(z+i)^{-1}.$$

In this case, the polynomial $P = 0$, as can be seen by direct computation of $f - P_i - P_{-i}$, or by noting that $f(z), P_i(z)$, and $P_{-i}(z)$ all tend to 0 as $z \to \infty$. We thus have

$$f(z) = \frac{1}{2}(z-i)^{-1} + \frac{1}{2}(z+i)^{-1}.$$