# THE RESIDUE THEOREM

**Definition.** Let $f$ be a function with an isolated singularity at $z_0$. The **residue of $f$ at $z_0$** is $\operatorname{Res}[f(z), z_0] = a_{-1}$, where the Laurent series expansion of $f$ at $z_0$ is

$$\sum_{k=-\infty}^{\infty} a_k (z - z_0)^k.$$

**Example.** We have

$$\operatorname{Res}[z^k, 0] = \begin{cases} 1 & \text{if } k = -1 \\ 0 & \text{if } k \neq -1 \end{cases}.$$

**Theorem** (The residue theorem)**.** *Let $D$ be a bounded domain with piecewise smooth boundary, and let $f$ be analytic on $\overline{D}$ except for finitely many isolated singularities $z_1, \ldots, z_n \in D$. Then*

$$\int_{\partial D} f(z)\, dz = \sum_{j=1}^{n} 2\pi i \operatorname{Res}[f(z), z_j].$$

*Proof.* For $j = 1, \ldots, n$, choose $\epsilon_j$ small enough so that the sets $\overline{B(z_j, \epsilon_j)}$ are pairwise disjoint and contained in $D$. Then $f$ is analytic on $D' = D \setminus \bigcup \overline{B(z_j, \epsilon_j)}$, so Cauchy's theorem gives

$$0 = \int_{\partial D'} f(z)\, dz = \int_{\partial D} f(z)\, dz - \sum_{j=1}^{n} \int_{\partial B(z_j, \epsilon_j)} f(z)\, dz = \int_{\partial D} f(z)\, dz - \sum_{j=1}^{n} 2\pi i \operatorname{Res}[f(z), z_j],$$

using that

$$\begin{aligned} \int_{\partial B(z_j, \epsilon_j)} f(z)\, dz &= \int_{\partial B(z_j, \epsilon_j)} \sum_{k=-\infty}^{\infty} a_k (z - z_j)^k\, dz = \sum_{k=-\infty}^{\infty} a_k \int_{\partial B(z_j, \epsilon_j)} (z - z_j)^k\, dz \\ &= 2\pi i a_{-1} = 2\pi i \operatorname{Res}[f(z), z_j], \end{aligned}$$

where, as usual, we may swap the sum and integral by uniform convergence of the series, and we use

$$\int_{\partial B(z_j, \epsilon_j)} (z - z_j)^k\, dz = \begin{cases} 2\pi i & \text{if } k = -1 \\ 0 & \text{if } k \neq -1. \end{cases}$$

□

**Example.** Let $f(z) = z/(1 + z^2) = \frac{1}{2}(z - i)^{-1} + \frac{1}{2}(z + i)^{-1}$, so $f$ has two isolated singularities at $i$ and $-i$, with

$$\operatorname{Res}[f(z), i] = \operatorname{Res}[f(z), -i] = \frac{1}{2}.$$

For $R > 0$, $R \neq 1$, we can apply the residue theorem to $B(0, R)$. We find

$$\oint_{|z|=R} f(z)\, dz = \begin{cases} 0 & \text{if } R < 1 \\ 2\pi i(\operatorname{Res}[f(z), i] + \operatorname{Res}[f(z), -i]) = 2\pi i & \text{if } R > 1. \end{cases}$$

**Computing residues, 4 tricks.** The following are some "rules" or tricks to compute residues of functions without having to compute the Laurent expansions directly. Recall that a pole is called **simple** if it has order 1 and **double** if it has order 2.

**1:** If $f$ has a simple pole at $z_0$, then $\operatorname{Res}[f(z), z_0] = \lim_{z\to z_0}(z-z_0)f(z)$.

**2:** If $f$ has a double (or simple) pole at $z_0$, then $\operatorname{Res}[f(z), z_0] = \lim_{z\to z_0}\frac{d}{dz}\left[(z-z_0)^2 f(z)\right]$.

**3:** If $f = g/h$ where $g, h$ are analytic at $z_0$ and $h$ has a simple zero at $z_0$, then $\operatorname{Res}[f(z), z_0] = g(z_0)/h'(z_0)$.[1]

**4:** If $g$ has a simple zero at $z_0$, then $\operatorname{Res}[1/g, z_0] = 1/g'(z_0)$.

*Proof of these.* We assume for simplicity that $z_0 = 0$.

**1:** If $f$ has a simple pole at 0, then near 0, $f(z) = a_{-1}z^{-1} + a_0 + \dots$, so

$$\lim_{z\to 0} zf(z) = \lim_{z\to 0}(a_{-1} + a_0 z + \dots) = a_{-1}.$$

**2:** Near 0, $f$ has the form $f(z) = a_{-2}z^{-2} + a_{-1}z^{-1} + \dots$, so

$$\frac{d}{dz}\left[z^2 f(z)\right] = \frac{d}{dz}\left(a_{-2} + a_{-1}z + \dots\right) = a_{-1} + 2a_0 z + \dots.$$

As $z \to 0$, this tends to $a_{-1}$.

**3:** Since $h$ has a simple zero, we can write $h(z) = z\widetilde{h}(z)$ for some analytic function $\widetilde{h}$ with $\widetilde{h}(0) \neq 0$. Then $h'(0) = \widetilde{h}(0)$, so, by **1**,

$$\operatorname{Res}[f(z), 0] = \lim_{z\to 0}\frac{zg(z)}{h(z)} = \lim_{z\to 0}\frac{g(z)}{\widetilde{h}(z)} = \frac{g(0)}{\widetilde{h}(0)} = \frac{g(0)}{h'(0)}.$$

**4:** Apply **3** with numerator 1 and denominator $g$. □

**Example.** The function $\frac{\cos z}{z}$ has a simple pole at 0, and we find the residue at 0 by trick 1 as

$$\operatorname{Res}\left[\frac{\cos z}{z}, 0\right] = \lim_{z\to 0} z\left(\frac{\cos z}{z}\right) = \lim_{z\to 0}\cos z = \cos(0) = 1.$$

**Example.** The function $\frac{\cos z}{z^2}$ has a double pole at $z = 0$, and we find the residue at 0 by trick 2 as

$$\operatorname{Res}\left[\frac{\cos z}{z^2}, 0\right] = \lim_{z\to 0}\frac{d}{dz}\left[z^2\left(\frac{\cos z}{z^2}\right)\right] = \lim_{z\to 0}\frac{d}{dz}\left[\cos z\right] = \lim_{z\to 0}(-\sin z) = -\sin(0) = 0.$$

**Example.** The function $\tan z = \frac{\sin z}{\cos z}$ has a simple pole at $z = \frac{\pi}{2}$, where we find the residue by trick 3 as

$$\operatorname{Res}\left[\tan z, \frac{\pi}{2}\right] = \frac{\sin\left(\frac{\pi}{2}\right)}{\cos'\left(\frac{\pi}{2}\right)} = \frac{\sin\left(\frac{\pi}{2}\right)}{-\sin\left(\frac{\pi}{2}\right)} = -1.$$

**Definite integrals via residues.** Consider the function $f(z) = \frac{1}{1+z^2}$. This has a partial fraction decomposition

$$\frac{1}{1+z^2} = \frac{i}{2}\frac{1}{z+i} - \frac{i}{2}\frac{1}{z-i},$$

and in particular we have

$$\operatorname{Res}[f, i] = -\frac{i}{2}, \quad \operatorname{Res}[f, -i] = \frac{i}{2}.$$

[1]Since $h$ has a simple zero at $z_0$, $h'(z_0) \neq 0$.

Consider now the real integral[2]

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{R\to\infty}\int_{-R}^{R} \frac{1}{1+x^2}\,dx.$$

Now for each $R$, let $C_R$ denote the path tracing a semicircle from $R$ through $iR$ to $-R$ counterclockwise, and let $\gamma_R$ denote the curve consisting of a line segment from $-R$ to $R$, followed by $C_R$. We then have

$$\int_{\gamma_R} \frac{1}{1+z^2}\,dz = \int_{-R}^{R} \frac{1}{1+x^2}\,dx + \int_{C_R} \frac{1}{1+z^2}\,dz.$$

The left-hand side is evaluated by the residue theorem. For $R > 1$, the only singularity of $f$ inside the domain bounded by $\gamma_R$ is at $i$, and so we have

$$\int_{\gamma_R} \frac{1}{1+z^2}\,dz = 2\pi i \operatorname{Res}\left[\frac{1}{1+z^2}, i\right] = 2\pi i\left(-\frac{i}{2}\right) = \pi.$$

On the other hand, for $|z| = R$ (and $R > 1$), we have

$$\left|\frac{1}{1+z^2}\right| \le \frac{1}{|z^2|-1} = \frac{1}{|z|^2-1} = \frac{1}{R^2-1},$$

and so the $ML$-estimate gives (with the length $L$ of $C_R$ equal to $\pi R$) that

$$\left|\int_{C_R} \frac{1}{1+z^2}\,dz\right| \le ML = \frac{1}{R^2-1}(\pi R) = \frac{\pi}{R-R^{-1}} \to 0$$

as $R \to \infty$. Putting this all together gives

$$\begin{aligned}\pi &= \lim_{R\to\infty}\int_{\gamma_R} \frac{1}{1+z^2}\,dz = \lim_{R\to\infty}\int_{-R}^{R} \frac{1}{1+x^2}\,dx + \lim_{R\to\infty}\int_{C_R} \frac{1}{1+z^2}\,dz \\ &= \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx + 0,\end{aligned}$$

and so

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi.$$

The method generalises without problem to a whole class of integrals,[3] to give the following:

**Proposition.** *Let $f(z) = \frac{p(z)}{q(z)}$ be a rational function with $\deg p \le \deg q - 2$, and with $q(x) \neq 0$ for all $x \in \mathbb{R}$. Then*

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i S,$$

*where $S$ is the sum of all residues of $f$ in the upper half-plane $\{x + iy \mid y > 0\}$.*

*Proof.* Integrate as before over the path $\gamma_R$. The assumption $q(x) \neq 0$ for all $x \in \mathbb{R}$ means we can apply the residue theorem, and when $R \gg 0$, all the poles of $f$ in the upper half-plane are within the domain bounded by $\gamma_R$. The residue theorem then gives $\int_{\gamma_R} f(z)\,dz = 2\pi i S$. The assumption $\deg p \le \deg q - 2$ gives that

[2] This particular integral can also be evaluated by using that $\arctan(x)$ is an antiderivative for $\frac{1}{1+x^2}$, but the method we present here is much more general.

[3] I recommend not memorising this theorem, but thinking about why it holds.

$R \max_{|z|=R} |f(z)| \to 0$ as $R \to \infty$, and so the $ML$-estimate gives $\int_{C_R} f(z)\,dz \to 0$ as $R \to \infty$. $\square$