

UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT2410 — Introduction to Complex Analysis

Day of examination: Wednesday 10. December 2014

Examination hours: 14.30 – 18.30

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (8 marks)

Consider the function $f : \mathbb{C} \rightarrow \mathbb{C}$ given by $f(x, y) = e^x \cos(y) - ie^x \sin(y)$.

(a) Calculate $\frac{\partial f}{\partial z}$.

(b) Calculate $\frac{\partial f}{\partial \bar{z}}$.

(c) Is f a holomorphic function? Justify your answer.

Problem 2 (12 marks)

Let f be an entire function. Then f has a power series representation $f(z) = \sum_{n=0}^{\infty} a_n z^n$ that converges for all $z \in \mathbb{C}$. Suppose that there exists a real constant $C > 0$ such that

$$|f(z)| \leq C|z|^2 \quad \text{for all } z \in \mathbb{C} \text{ such that } |z| \geq 1.$$

Prove that f is a polynomial of degree less than or equal to 2, and that $|a_2| \leq C$.

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Problem 3 (16 marks)

Consider the function $f(z) = \frac{1+z^2}{z} = \frac{1}{z} + z$.

- (a) Find a primitive for f on the slit plane $\mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Re}(z) \leq 0, \operatorname{Im}(z) = 0\}$.
- (b) Let $C_1 = \{z \in \mathbb{C} : |z| = 1, \operatorname{Re}(z) \geq 0\}$ be the semicircle in the right half plane, starting at the point i and ending at the point $-i$. Sketch C_1 and use your primitive for f from part (a) to compute

$$\int_{C_1} f(z) dz.$$

- (c) Let $C_2 = \{z \in \mathbb{C} : |z| = 1, \operatorname{Re}(z) \leq 0\}$ be the semicircle in the left half plane, starting at the point i and ending at the point $-i$. Sketch C_2 and use an explicit parameterisation of C_2 to compute

$$\int_{C_2} f(z) dz.$$

- (d) Let $C = \{z \in \mathbb{C} : |z| = 1\}$ be the circle of radius 1, centred at 0, equipped with the positive (counterclockwise) orientation. Using your results from parts (b) and (c), find

$$\int_C f(z) dz.$$

- (e) Does f have a primitive on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$? Justify your answer.

Problem 4 (16 marks)

Consider the holomorphic function $f : \mathbb{C} \setminus \{0, \frac{1}{3}\} \rightarrow \mathbb{C}$ given by the formula

$$f(z) = \frac{3+z^2}{z^2} - \frac{8}{3z-1}.$$

- (a) Find all zeros of f and determine their orders.
- (b) Classify all isolated singularities of f and find the residue at each pole.
- (c) Determine what sort of singularity f has at infinity.
- (d) Let γ be a circle of radius 2, centred at 0, equipped with the positive (counterclockwise) orientation. Compute the integral

$$\int_{\gamma} f(z) dz.$$

Problem 5 (16 marks)

Consider the entire function $f(z) = z^n + 2^n$, where $n \geq 1$ is an integer.

- (a) Find all zeros of f .
- (b) Prove that $|f(z)| \geq 2^n - 1$ for all $z \in \mathbb{C}$ such that $|z| = 1$.
- (c) Let γ be the circle of radius 1, centred at 0, equipped with the positive (counterclockwise) orientation. Using Rouché's theorem and the argument principle, compute the integral

$$\int_{\gamma} \frac{nz^{n-1} + 2(n-1)z^{n-2} + 2^2(n-2)z^{n-3} + \dots + 2^{n-1}}{z^n + 2z^{n-1} + 2^2z^{n-2} + \dots + 2^{n-1}z + 2^n} dz.$$

Problem 6 (16 marks)

Let $\mathbb{P} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere. Consider the Möbius transformation $T \in \text{Aut}(\mathbb{P})$ given by the formula

$$T(z) = \frac{(-2+i)z+6}{z+3+i}.$$

- (a) Find the values $T(0)$, $T(1)$, $T(\infty)$.
- (b) Find all fixed points of T . (Recall that $z_0 \in \mathbb{P}$ is a fixed point of T if $T(z_0) = z_0$.)
- (c) Let $S \in \text{Aut}(\mathbb{P})$ be a Möbius transformation such that

$$S\left(\frac{9-3i}{5}\right) = 0, \quad S(1) = 1, \quad S(-2+i) = \infty.$$

Prove that $S(-6) = -6$.

Problem 7 (16 marks)

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ be the open upper half plane, and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be the complex plane with the origin removed.

- (a) Prove that every holomorphic function $f : \mathbb{C} \rightarrow \mathbb{H}$ is constant.
(Hint: You may use the fact that there exists a biholomorphism $\varphi : \mathbb{H} \rightarrow \mathbb{D}$, where $\mathbb{D}$ is the open unit disc.)
- (b) Let $g : \mathbb{C}^* \rightarrow \mathbb{H}$ be a holomorphic function. Using the result of part (a), or otherwise, prove that g is constant.
(Hint: Consider $h(z) = g(\exp(z))$, where $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ is the complex exponential.)

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