

1. $f: \mathbb{C} \rightarrow \mathbb{C}$, $f(x, y) = e^x \cos y - i e^x \sin y$

$$(a) \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right)$$

$$= \frac{1}{2} \left(e^x \cos y - i e^x \sin y - i (-e^x \sin y - i e^x \cos y) \right) \\ = 0$$

$$(b) \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} \right)$$

$$= \frac{1}{2} \left(e^x \cos y - i e^x \sin y + i (-e^x \sin y - i e^x \cos y) \right) \\ = e^x \cos y - i e^x \sin y = f(x, y)$$

(c) If f is holomorphic on $\mathbb{C}$ then

$$\frac{\partial f}{\partial \bar{z}}(x, y) = 0 \quad \text{for all } (x, y) \in \mathbb{C}.$$

Since $\frac{\partial f}{\partial \bar{z}}$ does not vanish on all of $\mathbb{C}$,

f is not holomorphic.

2. $f: \mathbb{C} \rightarrow \mathbb{C}$ entire, $f(z) = \sum a_n z^n$ for all $z \in \mathbb{C}$.

Note that $f^{(n)}(0) = n! a_n$.

By Cauchy inequalities,

$$|f^{(n)}(0)| \leq \frac{n! \|f\|_{C_R}}{R^n} \quad \text{where } C_R \text{ is the circle of radius } R \text{ centred at } 0.$$

$$\text{Let } R \geq 1, \text{ then } \|f\|_{C_R} = \max_{z \in C_R} |f(z)| \leq \max_{z \in C_R} C |z|^2 = CR^2$$

$$\text{So } |f^{(n)}(0)| \leq \frac{n! CR^2}{R^n} = \frac{n! C}{R^{n-2}}.$$

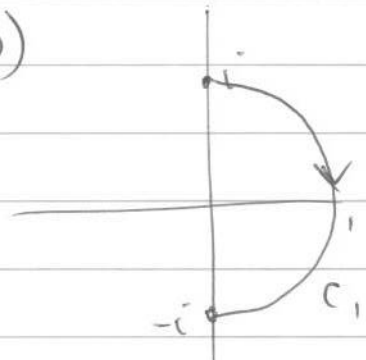
If $n \geq 3$, let $R \rightarrow \infty$ and then $|f^{(n)}(0)| = 0$
 $\Rightarrow a_n = 0$ for all $n \geq 3$, so $f(z) = a_0 + a_1 z + a_2 z^2$.

Furthermore, taking $R = 2$ gives

$$|2! a_2| = |f^{(2)}(0)| \leq \frac{2! C}{2^{2-2}} \Rightarrow |a_2| \leq C.$$

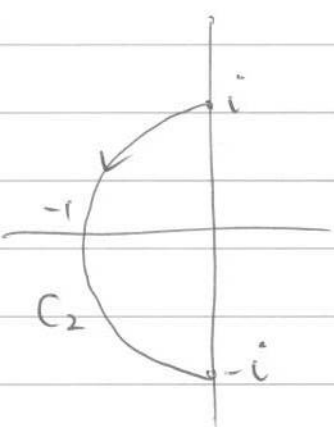
3. $f(z) = \frac{1+z^2}{z} = \frac{1}{z} + z$

(a) $F(z) = \text{Log } z + \frac{1}{2} z^2$ is a primitive for f on the slit plane, where $\text{Log}(z) = \log|z| + i\theta$, for $z = |z|e^{i\theta}$, $\theta \in (-\pi, \pi)$ is the principal branch of the logarithm.

(b)  $\int_{C_1} f dz = F(-i) - F(i)$

$$= \text{Log}(-i) + \frac{1}{2}(-i)^2 - \text{Log}(i) - \frac{1}{2}(i)^2$$

$$= -\frac{\pi}{2}i - \frac{1}{2} - i\frac{\pi}{2} + \frac{1}{2} = \underline{-\pi i}$$

(c)  Let $z(\theta) = e^{i\theta}$, $\theta = \pi/2$ to $3\pi/2$.
 $z'(\theta) = ie^{i\theta}$

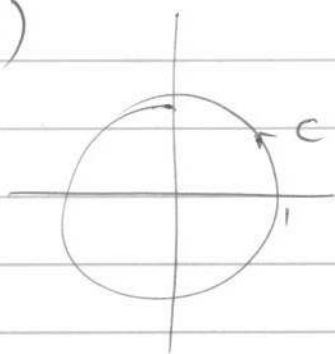
$$\int_{C_2} f dz = \int_{\pi/2}^{3\pi/2} \left(\frac{1}{e^{i\theta}} + e^{i\theta} \right) \cdot ie^{i\theta} d\theta$$

$$= i \int_{\pi/2}^{3\pi/2} (1 + e^{2i\theta}) d\theta = i \left(\theta + \frac{1}{2i} e^{2i\theta} \right) \Big|_{\pi/2}^{3\pi/2}$$

$$= i \frac{3\pi}{2} - i \frac{\pi}{2} + \frac{1}{2} e^{3\pi i} - \frac{1}{2} e^{i\pi}$$

$$= \underline{i\pi}$$

(d)



$$\begin{aligned}\int_C f dz &= -\int_{C_1} f dz + \int_{C_2} f dz \\ &= i\pi + i\pi = \underline{2\pi i}\end{aligned}$$

e) If f had a primitive on $\mathbb{C}^*$ then the integral of f around any closed loop in $\mathbb{C}^*$ would be zero. But C is a closed loop in $\mathbb{C}^*$ and $\int_C f dz \neq 0$, hence f cannot have a primitive on $\mathbb{C}^*$.

4/ $f: \mathbb{C} \setminus \{0, \frac{1}{3}\} \rightarrow \mathbb{C}$, $f(z) = \frac{3+z^2}{z^2} - \frac{8}{3z-1}$

(a) $f(z) = \frac{(3+z^2)(3z-1) - 8z^2}{z^2(3z-1)} = \frac{3z^3 - 9z^2 + 9z - 3}{z^2(3z-1)}$

$= \frac{(z-1)^3}{z^2(z-\frac{1}{3})}$ so f has a zero at $z=1$ of order 3.

(b) f has a pole at $z=0$ of order 2

f has a pole at $z=\frac{1}{3}$ of order 1 (simple pole).

$\text{res}_{\frac{1}{3}} f = \lim_{z \rightarrow \frac{1}{3}} (z - \frac{1}{3}) f(z) = \frac{(\frac{1}{3} - 1)^3}{(\frac{1}{3})^2} = \frac{-\frac{8}{27}}{1/9} = -\frac{8}{3}$

$\text{res}_0 f = \lim_{z \rightarrow 0} \frac{d}{dz} z^2 f(z) = \lim_{z \rightarrow 0} \frac{3(z-1)^2(z-\frac{1}{3}) - (z-1)^3}{(z-\frac{1}{3})^2}$

$= \frac{3 \cdot (-\frac{1}{3}) - (-1)}{(-\frac{1}{3})^2} = 0.$

(c) $f(\frac{1}{z}) = \frac{(\frac{1}{z} - 1)^3}{\frac{1}{z^2}(\frac{1}{z} - \frac{1}{3})} \cdot \frac{z^3}{z^3} = \frac{(1-z)^3}{(1-\frac{z}{3})}$

Let $z \rightarrow 0$, we see that $f(\frac{1}{z}) \rightarrow 1$,

so $f(\frac{1}{z})$ has a removable singularity at 0 .

Thus $f(z)$ has a removable singularity at ∞ .

(d) $\int_{\gamma} f(z) dz = 2\pi i \sum \text{res}$ where ~~all~~ residues are summed at all poles inside γ .

$$= 2\pi i (0 - 8/3) = -\frac{16\pi i}{3}$$

5. $f(z) = z^n + 2^n \quad n \geq 1.$

(a) $f(z) = 0 \Leftrightarrow z^n = -2^n = 2^n e^{i\pi + 2k\pi}$
 $\Leftrightarrow z = 2 e^{i\pi/n + 2k\pi/n} \quad k = 0, 1, 2, \dots, n-1.$

(b) $|f(z)| = |z^n + 2^n| \geq ||z^n| - |2^n||$
 $= |1 - 2^n| \quad \text{since } |z| = 1$
 $= 2^n - 1 \quad \text{since } n \geq 1.$

(c) let $g(z) = 2z^{n-1} + 2^2 z^{n-2} + \dots + 2^{n-1} z$,
 $h(z) = f(z) + g(z)$, then the given
integral is $\int_{\gamma} \frac{h'(z)}{h(z)} = 2\pi i (\# \text{zeros of } h \text{ inside } \gamma - \# \text{poles of } h \text{ inside } \gamma)$
by the argument principle.
 h is holomorphic therefore has no poles.

Now $|g(z)| \leq 2|z^{n-1}| + 2^2|z^{n-2}| + \dots + 2^{n-1}|z|$ by triangle inequality
 $= 2 + 2^2 + \dots + 2^{n-1} \quad \text{for } |z| = 1$
 $= 2^n - 2.$

So $|g(z)| \leq 2^n - 2 < 2^n - 1 \leq |f(z)|$ on circle $|z| = 1$.
By Rouché's theorem, $gh = f + g$ has the same number
of zeros inside γ as f . But all zeros
of f have modulus 2 (part (a)).

So $\int_{\gamma} \frac{h'(z)}{h(z)} = 2\pi i (\cancel{0} - 0) = 0.$

$$6. \quad T(z) = \frac{(-2+i)z + 6}{z + 3 + i}$$

$$(a) \quad T(0) = \frac{6}{3+i} \cdot \frac{3-i}{3-i} = \frac{18-6i}{10} = \frac{9-3i}{5}$$

$$T(1) = \frac{-2+i+6}{1+3+i} = \frac{4+i}{4+i} = 1$$

$$T(\infty) = \lim_{z \rightarrow \infty} T\left(\frac{1}{z}\right) = \lim_{z \rightarrow \infty} \frac{(-2+i)\frac{1}{z} + 6}{\frac{1}{z} + 3 + i} \cdot \frac{z}{z}$$

$$= \lim_{z \rightarrow \infty} \frac{(-2+i) + 6z}{1 + (3+i)z} = -2+i$$

(b) We already know 1 is a fixed point by (a).
 solve: $T(z) = z = \frac{(-2+i)z + 6}{z + 3 + i}$

$$\Leftrightarrow z^2 + (3+i)z = (-2+i)z + 6$$

$$\Leftrightarrow z^2 + 5z + 6 = 0$$

$$\Leftrightarrow (z+6)(z-1) = 0$$

so $z=6$ is the other fixed point.

$$\text{Check: } T(6) = \frac{(-2+i)(6) + 6}{-6+3+i} = \frac{12-6i+6}{-3+i} = \frac{18-6i}{-3+i}$$

$$= -6 \quad \checkmark$$

$$(c) \quad S\left(\frac{9-3i}{5}\right) = 0, \quad S(1) = 1, \quad S(-2+i) = \infty.$$

$$\text{Then } T \circ S\left(\frac{9-3i}{5}\right) = T(0) = \frac{9-3i}{5}$$

$$\cancel{TS} \quad T \circ S(1) = T(1) = 1$$

$$T \circ S(-2+i) = T(\infty) = -2+i$$

$$\text{So } T \circ S(z) = z \quad \text{for } z = \frac{9-3i}{5}, 1, -2+i$$

That is, $T \circ S = \text{id}$ at 3 distinct points in $\mathbb{P}$. But if two Möbius transformations agree at 3 points then they agree everywhere. Hence $T \circ S = \text{id}$ everywhere, i.e. $T \circ S(z) = z \quad \forall z \in \mathbb{P}$.

$$\Rightarrow S(z) = T^{-1}(z) \quad \forall z \in \mathbb{P}.$$

$$\text{Then } S(-6) = T^{-1}(-6) = -6$$

(since $T(6) = 6$ is a fixed point,
so $6 = T^{-1}(6)$ also is a fixed
point for T^{-1}).

7(a) let $f: \mathbb{C} \rightarrow \mathbb{H}$ be hol. let $\varphi: \mathbb{H} \rightarrow \mathbb{D}$ be a bihol.

~~7(a)~~ Then $\varphi \circ f: \mathbb{C} \rightarrow \mathbb{D}$ is a bounded hol. function hence is constant by Liouville's theorem. Then $\varphi^{-1} \circ (\varphi \circ f) = f$ must also be constant. (Recall φ^{-1} exists since φ is a biholomorphism.)

b) let $g: \mathbb{C}^* \rightarrow \mathbb{H}$ be hol.

Then $h(z) = g \circ \exp(z)$, $z \in \mathbb{C}$, is a holomorphic function and $h: \mathbb{C} \rightarrow \mathbb{H}$.

By part (a) h is constant.

That is, there exists some value $c \in \mathbb{H}$ s.t. $h(z) = g(\exp(z)) = c$ for all $z \in \mathbb{C}$.

let $w \in \mathbb{C}^*$ be arbitrary. Then $\exists z \in \mathbb{C}$ s.t. $\exp(z) = w$ ($\exp$ is surjective).

Hence

$$g(w) = g(\exp(z)) = h(z) = c.$$

w was arbitrary, so $g(w) = c$ for all $w \in \mathbb{C}^*$.

Thus g is constant.