

Exam 2015 Solution

①

a) We see that the poles are at 0 and -2.

To find the order, we will first find the zeros of $f(z)$.

$$f(z)=0 \Leftrightarrow \frac{z^2+z-2}{z^2(z+2)^2}=0 \Rightarrow z^2+z-2=0 \\ \Rightarrow z=1 \text{ or } z=-2.$$

$$\text{So } f(z) = \frac{(z-1)(z+2)}{z^2(z+2)^2} = \frac{z-1}{z^2(z+2)}.$$

We have a pole at 0 of order 2 $\left(f(z) = z^{-2} \left(\frac{z-1}{z+2}\right)\right)$
and a pole at -2 of order 1 $\left(f(z) = (z+2)^{-1} \left(\frac{z-1}{z^2}\right)\right)$

$$\text{b) } \operatorname{res}_{-2} f(z) = \lim_{z \rightarrow -2} (z+2) f(z) = \lim_{z \rightarrow -2} \frac{z-1}{z^2} = \frac{-2-1}{(-2)^2} = \frac{-3}{4}$$

$$\operatorname{res}_0 f(z) = \lim_{z \rightarrow 0} \frac{\partial}{\partial z} z^2 f(z) = \lim_{z \rightarrow 0} \frac{\partial}{\partial z} \frac{z-1}{z+2} \\ = \lim_{z \rightarrow 0} \frac{1 \cdot (z+2) - (z-1) \cdot 1}{(z+2)^2} = \frac{3}{(0+2)^2} = \frac{3}{4}$$

c)

$$\int_{\gamma} f(z) dz = 2\pi i \cdot \operatorname{res}_0 f(z) = \frac{6\pi i}{4} = \frac{3\pi i}{2}$$

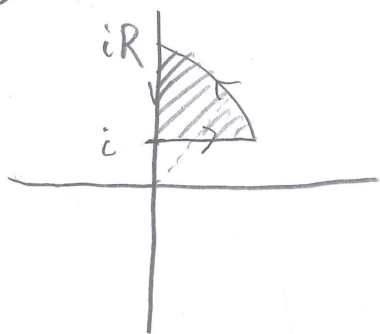
as -2 is outside of γ .

②

a)

$$\begin{aligned} f(z) &= \frac{1}{(t+i)^2} = \frac{(t-i)^2}{(t+i)^2(t-i)^2} = \frac{t^2 - 2it - 1}{(t+i)(t-i)^2} \\ &= \frac{t^2 - 1 - 2it}{(t^2 + 1)^2} = \frac{t^2 - 1}{(t^2 + 1)^2} + i \frac{-2t}{(t^2 + 1)^2} \\ &= u(t) + i v(t) \quad \text{with} \quad u(t) = \frac{t^2 - 1}{(t^2 + 1)^2}, \quad v(t) = \frac{-2t}{(t^2 + 1)^2} \end{aligned}$$

b)



As $f(z)$ is holomorphic on Ω_R ,

we have

$$\int_{\gamma_R} f(z) dz = 0.$$

c)

We have

$$\left| \int_{C_R} f(z) dz \right| \leq l(C_R) \cdot \sup_{z \in C_R} \left| \frac{1}{z^2} \right| = l(C_R) \cdot \frac{1}{R^2},$$

as $|z| = R$ for $z \in C_R$. Here, $l(C_R)$ is the length of

C_R . We have $l(C_R) \leq \frac{\pi}{2} R$, so we have

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{\pi}{2} \frac{R}{R^2} = \frac{\pi}{2R} \rightarrow 0 \text{ as } R \rightarrow \infty.$$

$$\text{So } \lim_{R \rightarrow \infty} \left| \int_{C_R} f(z) dz \right| \leq 0 \Rightarrow \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0 \Rightarrow \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0.$$

d) We have $\int_{\gamma_R} f(z) dz = \int_0^r f(\alpha(t)) \alpha'(t) dt + \int_{C_R} f(z) dz - \int_1^R f(it) \cdot i dt$

where r is given by  $r = \sqrt{R^2 - 1}$

We have $\int_0^r f(\alpha(t)) \cdot \alpha'(t) dt = \int_0^r u(t) dt + i \int_0^r v(t) dt$
 $= \int_0^r \frac{x^2 - 1}{(x^2 + 1)^2} dx + i \int_0^r \frac{-2x}{(x^2 + 1)^2} dx$

and $\int_1^R f(it) i dt = i \int_1^R \frac{-1}{t} dt$

So we see that $\operatorname{Re} \left(\int_{\gamma_R} f(z) dz \right) = \int_0^r \frac{x^2 - 1}{(x^2 + 1)^2} dx + \operatorname{Re} \left(\int_{C_R} f(z) dz \right)$

Giving us $\int_0^\infty \frac{x^2 - 1}{(x^2 + 1)^2} dx = \lim_{R \rightarrow \infty} \int_0^{\sqrt{R^2 - 1}} \frac{x^2 - 1}{(x^2 + 1)^2} dx = \lim_{R \rightarrow \infty} \left(\operatorname{Re} \left(\int_{\gamma_R} f(z) dz \right) - \int_{C_R} f(z) dz \right)$

$= \operatorname{Re} \left(\lim_{R \rightarrow \infty} 0 - \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz \right)$

$= \operatorname{Re}(0 - 0) = \operatorname{Re}(0) = \underline{0}$

So $\int_0^\infty \frac{x^2 - 1}{(x^2 + 1)^2} dx = 0$

③

Let $f(z) = -4z^3$, $g(z) = z^7 + z - 1$.

Then $|f(z)| = |-4z^3| = 4|z|^3 = 4$ on the circle $|z|=1$
and $|g(z)| = |z^7 + z - 1| \leq |z|^7 + |z| + |-1| = 3$ on the circle $|z|=1$

So $|g(z)| < |f(z)|$ on the circle $|z|=1$, which means by

Rouché's Thm that $f(z)$ and $f(z) + g(z) = z^7 - 4z^3 + z - 1$
have the same number of zeros in the disc $|z| < 1$.

And as $f(z)$ has 3 zeros (0, multiplicity 3), we have that

$z^7 - 4z^3 + z - 1$ have 3 zeros as well.

④

a) $f(\infty) = \frac{\infty - 1}{\infty + 1} = \lim_{z \rightarrow \infty} \frac{z-1}{z+1} = \frac{1}{1} = 1$

$f(z) = z \Leftrightarrow \frac{z-1}{z+1} = z \Leftrightarrow z-1 = z^2 + z$

$\Leftrightarrow z^2 = -1 \Leftrightarrow z = \pm i$.

So i and $-i$ are the fixed points for f .

b) We see that $g(f(-i)) = g(-i) = -i$, $g(f(i)) = g(i) = i$ and
 $g(f(\infty)) = g(1) = \infty$, so we must have $g = f^{-1}$.

Corresponding matrix of f is $M = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

The inverse is $M^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, so we have

$g(z) = \frac{\frac{1}{2}z + \frac{1}{2}}{-\frac{1}{2}z + \frac{1}{2}} = \frac{z+1}{1-z}$