UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in:	MAT2440 — Differential equations and optimal control theory
Day of examination:	2 June 2017
Examination hours:	14.30 – 18.30

This problem set consists of 3 pages.

Appendices:	None
Permitted aids:	None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Let A be the matrix

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}.$$

a

Determine $\exp(tA)$.

b

Let $b(t)$ be given as

$$b(t) = \begin{pmatrix} e^{-t} \\ 0 \\ 0 \end{pmatrix}.$$

Find the solution of the differential equation

$$\dot{x} = Ax + b$$

with $x(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$.

Problem 2

Consider the optimalization problem

$$J(x) = \int_0^1 (2x^2 + 2x\dot{x} + \dot{x}^2)dt \qquad x(0) = 0 \quad x(1) = 1.$$

(Continued on page 2.)

a

Determine the corresponding Euler-Lagrange equation and find the general solution of it.

b

Determine a function $x(t)$ with $x(0) = 0$ and $x(1) = 1$ such that $J(x)$ is minimal.

Problem 3

a

Let $\alpha > 0$. Show that $f(x) = x^{1/2}$ is Lipschitz in the interval $[\alpha, \infty)$ and give a Lipschitz constant. Is $f(x)$ Lipschitz in any interval of type $[0, \beta]$?

b

Consider the differential equation

$$\dot{x} = 2x^{1/2} \tag{*}$$

in the region $E = \{\, x \in \mathbb{R} \mid x > 0 \,\}$. Determine the general solution of the equation and for each $t_0 \in \mathbb{R}$ and each $x_0 \in E$ a solution with $x(t_0) = x_0$. Give an argument for why it is unique.

c

Let now $F = \{\, x \in \mathbb{R} \mid x \geq 0 \,\}$. Show that for every t_0 there are at least two solutions of (*) in F with $x(t_0) = 0$ and that both are determined for all t.

Problem 4

Consider the non-linear system

$$\begin{aligned} \dot{x} &= y - yx \\ \dot{y} &= x + yx. \end{aligned} \tag{**}$$

a

Show that the equilibria of the system (**) are $(0, 0)$ and $(1, -1)$, and determine the linearization of the system in each of these. Determine the type of linearizations.

b

Let $z(t) = (x(t), y(t))$ be a solution of the system (**). Show that $x(t) + y(t) = Ae^t$ for a suitable constant A. Show that if $z(0)$ lies on the line L given by $y + x = 0$, then $z(t)$ lies there for all t.

(Continued on page 3.)

c

Let (a, b) be a point *not* lying on the line L, and let $z(t)$ be the solution with $z(0) = (a, b)$. Show that $z(t)$ is not bounded when $t \to \infty$.

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