OBLIGATORY PROBLEMS IN MAT 2440

Deadline April 10, 2014, 2:30 p.m.

You need only answer one of the two problems 4 and 5. If you answer both, the best of your two solutions will count.

All the problems can be solved without the aid of a computer or a calculator, however you may also use any computer system (like Matlab, Mathematica, Maple, or the numpy package of Python). Whenever doing so, you should document the commands you are using and explain in detail how you apply the results.

PROBLEM 1

Let

$$P(x,y) = 5x^4y - y^5$$

and

$$Q(x,y) = x^5 - 5xy^4$$

Solve the differential equation

$$Pdx + Qdy = 0$$

(**a**) as an exact equation
(**b**) as a homogeneous equation.

PROBLEM 2

Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$$

Find a general solution of the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$

(**a**) by using the method of elimination (write the equation as a system of linear first-order equations),

(**b**) by the eigenvalue method.

PROBLEM 3

Let

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 3 & -3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Find a general solution of the differemtial equation

$$\mathbf{x}' = \mathbf{A}\mathbf{x}$$

PROBLEM 4

You need only answer one of the two problems 4 and 5. If you answer both, the best of your two solutions will count.

Let

$$F(t, x, \dot{x}) = x^2 + \frac{1}{2}t(t-1)\dot{x}^2,$$

and consider the problem

$$(1) \qquad \min \int_2^3 F(t, x(t), \dot{x}(t))\,\mathrm{d}t, \quad x(2) = 0,\ x(3) = \ln\frac{25}{27}.$$

(**a**) Find the Euler equation (E) of the problem in (1).

(**b**) Show that (E) has a first degree polynomial x_1 as a solution.

(**c**) Use reduction of order (see EP Exercise 2.2.36) to find another solution x_2 of (E) such that $x_2(t) = v(t)x_1(t),\ t \in [2,3]$.

(**d**) What is the general solution of (E) ? Find the only solution x^* of (E) that satisfies the endpoint conditions in (1).

(**e**) Decide if x^* minimizes the integral in (1).

PROBLEM 5

You need only answer one of the two problems 4 and 5. If you answer both, the best of your two solutions will count.

In the followig we shall study maxima and minima of

$$(*) \qquad J(x) = \int_1^2 F(t, x(t), \dot{x}(t))\,\mathrm{d}t, \qquad x(1) = -\frac{1}{3},\ x(2) \text{ is free .}$$

Here

$$F(t,x,\dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{6}g(t)\dot{x}^3,$$

and g is a positive ($g > 0$) C^2-function on $[1,2]$.

(**a**) Find the Euler equation (E) of the problem. Express $\dot{x}$ in terms of g.

From now on we let

$$g(t) = \frac{1}{t^2}, \qquad \text{for all } t \in [1,2].$$

(**b**) Find the general solution of (E) in this special case.

(**c**) Show that (E) has exactly two solutions x_1 and x_2 which satisfy the endpoint conditions of $(*)$.

(**d**) Consider the two following convex sets of functions,

$$R_1 = \{x \,:\, x \in C^2[1,2], x(1) = -\frac{1}{3}, x(2) \text{ is free}, \dot{x}(t) > -1 \quad (\text{for all } t \in [1,2])\}$$

and

$$R_2 = \{x \,:\, x \in C^2[1,2], x(1) = -\frac{1}{3}, x(2) \text{ is free}, \dot{x}(t) < -1 \quad (\text{for all } t \in [1,2])\}$$

Decide if the function $(x,\dot{x}) \mapsto F(t,x,\dot{x}) \quad (t \in [1,2]$ arbitrary, fixed) is concave or convex on R_1 or on R_2. Does $J(x)$ have a maximum or a minimum on any of the two sets R_1, R_2 ? Find the maximum and the minimum of $J(x)$ on R_1 and on R_2 if they exist.

Remark. The Euler equation of Problem 5 (a) (with $y = \dot{x}$) is a simple special case of the Abel differential equation of the 2nd kind:

$$(1) \qquad (y+s)\dot{y} + (p + qy + ry^2) = 0,$$

where $p, q, r,$ and s are C^1-functions of a real variable on an open interval I. Such equations can always be reduced to the simpler form

$$(2) \qquad z\dot{z} + (P + Qz) = 0$$

via a substitution $y = \alpha + \beta z$, where α and β are C^1-functions on I. In fact, we may let

$$\alpha = -s, \beta = e^{-\int r(t)\,\mathrm{d}t}, P = (p - qs + rs^2)e^{2\int r(t)\,\mathrm{d}t}, Q = (q - 2rs - \dot{s})e^{\int r(t)\,\mathrm{d}t}.$$

The simple special case (E) of Problem 5 (a) may be solved without such a reduction. Abel studied equation (1) in the 1820-s and obtained (among many other results) the above mentioned reduction to (2). (Œuvres complétes de Niels Henrik Abel, Tome Second, p. 26-35.)

THE END