Consider the 1st order linear system

$$(1)\quad \begin{cases} x_1' = p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t) \\ x_2' = p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t) \\ \vdots \qquad \cdots \\ \vdots \\ x_n' = p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t) \end{cases}$$

Equivalently:

$$(2)\quad \vec{X}' = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix} = P(t)\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}; \quad P(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \cdots & & \cdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix}$$

(2) is called homogeneous if $f_1 = f_2 = \cdots = f_n = 0$-function

Otherwise (2) is nonhomogeneous

$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is a solution, of (2)

if x_i satisfies the i-th equation $x_i' = P_i(t)x_i + f_i(t)$

$1 \le i \le n$

<u>Theorem 1</u> (Existence and Uniqueness)

Let $b_1, \dots, b_n$ are given numbers. Suppose that $f_1, f_2, \dots, f_n$ and $p_{11}, p_{12}, \dots, p_{nn}$ are continuous on an open interval I with $a \in I$. Then the system in (2) (or in (1)) has a <u>unique solution</u> on I that satisfies the initial conditions

$$x_1(a) = b_1, \; x_2(a) = b_2, \dots, x_n(a) = b_n$$

(<u>Proof</u> is not part of the course. See the Appendix (EP) Thm. 2 and Thm. 4 (p. 695 and 697) for proofs.)

The Method of Elimination

This is the method we have used in the previous examples. It is useful for systems of small dimensions (2- or 3-dimensional). For larger systems matrix methods are usually more useful.

~~Read~~

Example: Solve:

$x' = 4x - 3y$, $y' = 6x - 7y$

$x(0) = 2$, $y(0) = -1$

Solution. The 1st eq. gives

$$\begin{aligned} x'' &= 4x' - 3y' \\ &= 4x' - 3(6x - 7y) \quad \text{(by 2nd eq.)} \\ &= 4x' - 18x + 21y \quad (\text{using } 3y = 4x - x') \\ &= 4x' - 18x + 7(4x - x') \\ &= -3x' + 10x \end{aligned}$$

or $x'' + 3x' - 10x = 0$

with char. roots

$$r = \frac{1}{2}\Big[-3 \pm \underbrace{\sqrt{9+40}}_{7}\Big] = \begin{cases} 2 \\ -5 \end{cases}$$

So

$$x(t) = \underline{A e^{-5t} + B e^{2t}}$$

$$y(t) = \frac{1}{3}(4x - x') = \frac{1}{3}\Big[4Ae^{-5t} + 4Be^{2t}$$

$$- (-5Ae^{-5t} + 2Be^{2t})\Big]$$

$$= \underline{3Ae^{-5t} + \frac{2}{3}Be^{2t}}$$

(A, B arbitrary constants)

(i) $x(0) = A + B = 2$ $\quad|(-3)$

(ii) $y(0) = 3A + \frac{2}{3}B = -1$

$$-3B + \frac{2}{3}B = -6 - 1 = -7$$

$$-\frac{7}{3}B = -7, \quad \underline{B = 3}$$

$$A = 2 - B = \underline{-1}$$

$$x(t) = -e^{-5t} + 3e^{2t}$$
$$y(t) = -3e^{-5t} + 2e^{2t}$$

is the unique solution.

Matrices and Linear Systems (homogeneous)

We have already seen that we can write such systems on the form

$$\vec{x}' = P(t)\vec{x}, \quad \vec{x}' = \frac{d\vec{x}}{dt} = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix}$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $P(t) = \begin{bmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \cdot & \cdot & \cdot \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{bmatrix}$

Theorem 1 (Superposition)

Let $\vec{x}_1, \dots, \vec{x}_n$ be n solutions of the linear differential eq.

$$\vec{x}' = P(t)\vec{x}$$

on an open interval $I \subset \mathbb{R}$.

Then $\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \dots + c_n\vec{x}_n(t)$

is also a solution, for any choice of constants $c_1, \dots, c_n$.

Proof. We know that

$$\vec{x}_i' = P(t)\vec{x}_i \quad \text{for } i = 1, 2, \dots, n$$

Hence

$$\begin{aligned}\vec{x}' &= c_1\vec{x}_1' + \dots + c_n\vec{x}_n' \\ &= c_1 P(t)\vec{x}_1 + \dots + c_n P(t)\vec{x}_n \\ &= P(t)[c_1\vec{x}_1 + \dots + c_n\vec{x}_n] \\ &= P(t)\vec{x}, \quad t \in I. \quad \blacksquare\end{aligned}$$