Generalized eigenvectors (April 5, 2016)

P : $n\times n$ matrix (real entries)

λ : an eigenvalue of P.

$p(t) = |P - tI| = (-1)^n |tI - P|$:
the characteristic polynomial of λ (and P).

λ has multiplicity k if
$(t-\lambda)^k$ is a factor of $p(t)$,
but $(t-\lambda)^{k+1}$ is not a factor of $p(t)$.

If λ has only $m < k$ linearly independent eigenvectors corresponding to λ, then

$$d = k - m$$

is called the <u>defect</u> of λ

The case $k=2$ and defect $d=1$.

Consider a system

$$\vec{x}\,' = P\vec{x} \quad (\text{on an open interval } I \subset \mathbb{R})$$

where P has an eigenvalue λ of multiplicity 2 and defect $d=1$.
Thus λ has only a one-dimensional eigenspace, spanned by an eigenvector $\vec{v}_1$ for λ.

Then
$$\vec{X}_1(t) = e^{\lambda t}\,\vec{v}_1 \quad , \quad t \in I ,$$

is a solution of
$$\vec{X}' = P\vec{X} .$$

We proceed to find a solution $\vec{X}_2(t)$ that is lin. independent of $\vec{X}_1(t)$.

We could try to find a solution
$$\vec{X}_2(t) = t\,e^{\lambda t}\,\vec{v}_2 \quad , \quad \vec{v}_2 \neq \vec{0}$$

Then
$$\vec{X}_2{}'(t) = e^{\lambda t}(\lambda t + 1)\vec{v}_2$$

and
$$P\vec{X}_2(t) = t\,e^{\lambda t}\,P\vec{v}_2$$

Hence
$$e^{\lambda t}(\lambda t+1)\vec{v}_2 = t e^{\lambda t} P\vec{v}_2$$
$$(\lambda t+1)\vec{v}_2 = t\,P\vec{v}_2$$
$$\lambda t\,\vec{v}_2 - tP\vec{v}_2 + \vec{v}_2 = 0 \quad (t \in I)$$

or
$$(\lambda\vec{v}_2 - P\vec{v}_2)t + \vec{v}_2 \equiv 0 \quad \text{on } I$$

Hence $\lambda\vec{v}_2 - P\vec{v}_2 = \vec{0}$ and $\vec{v}_2 = \vec{0}$, a contradiction!

Must try something else (use what we learned from (example) problem last time (Problem 5.6.1)

Let
$$\vec{x}_2(t) = te^{\lambda t}\vec{v}_1 + \vec{v}_2 \cdot e^{\lambda t}$$
$$= e^{\lambda t}(t\vec{v}_1 + \vec{v}_2), \quad t \in I.$$

Then
$$\vec{x}_2'(t) = e^{\lambda t}(\lambda(t\vec{v}_1 + \vec{v}_2) + \vec{v}_1)$$
$$= e^{\lambda t}((\lambda t + 1)\vec{v}_1 + \lambda\vec{v}_2)$$

and
$$P\vec{x}_2(t) = e^{\lambda t}(tP\vec{v}_1 + P\vec{v}_2)$$
$$= e^{\lambda t}(t\lambda\vec{v}_1 + P\vec{v}_2),$$

since $\vec{v}_1$ was an eigenvector for λ.

Thus
$$(\lambda t + 1)\vec{v}_1 + \lambda\vec{v}_2 = t\lambda\vec{v}_1 + P\vec{v}_2$$
$$P\vec{v}_2 = \vec{v}_1 + \lambda\vec{v}_2$$
$$\boxed{(P - \lambda I)\vec{v}_2 = \vec{v}_1} \quad \text{and} \quad P\vec{v}_1 = \lambda\vec{v}_1$$

In particular,
$$\underbrace{(P - \lambda I)^2\vec{v}_2}_{} = (P - \lambda I)\vec{v}_1 = \underbrace{\vec{0}}_{}$$

Algorithm for defective multiplicity 2 eigenvalues:

1. First find a nonzero solution of
$$(P-\lambda I)^2 \vec{v}_2 = \vec{0},$$
such that
$$(P-\lambda I)\vec{v}_2 = \vec{v}_1 \neq \vec{0}$$
and hence $\vec{v}_1$ is an eigenvector for λ.

2. Then form two lin. independent solutions
$$\vec{x}_1(t) = e^{\lambda t}\vec{v}_1$$
$$\vec{x}_2(t) = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2), \quad t \in I.$$
of $\vec{x}' = P\vec{x}$
corresponding to λ.

Example Consider
$$\vec{x}' = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}\vec{x}, \quad P = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix}.$$

We find first eigenvalues/eigenspaces:
$$0 = |\lambda I - P| = \begin{vmatrix} \lambda-1 & +3 \\ -3 & \lambda-7 \end{vmatrix} = (\lambda-1)(\lambda-7)+9$$
$$= \lambda^2 - 8\lambda + 16 = \underline{(\lambda-4)^2},$$
so $\lambda=4$ has multiplicity $\underline{m = 2}$

Now

$$P - 4I = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Hence

$\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ spans the eigenspace of $\lambda = 4$.

Hence

defect $= d = m - 1 = 2 - 1 = \underline{1}$

Therefore, we calculate first

$$(P - 4I)^2 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus $(P - 4I)^2 \vec{v}_2 = \vec{0}$

for any choice of $\vec{v}_2$. We take

$\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (which is linearly independent from $\vec{u}_1$!)

Then

$$\vec{v}_1 = (P - 4I)\vec{v}_2 = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} = (-3)\vec{u}_1.$$

Hence $\vec{X}_2(t) = e^{4t}(t\vec{v}_1 + \vec{v}_2)$

is a solution (and is lin. independent of $\vec{X}_1(t) = e^{4t}\vec{v}_1$)

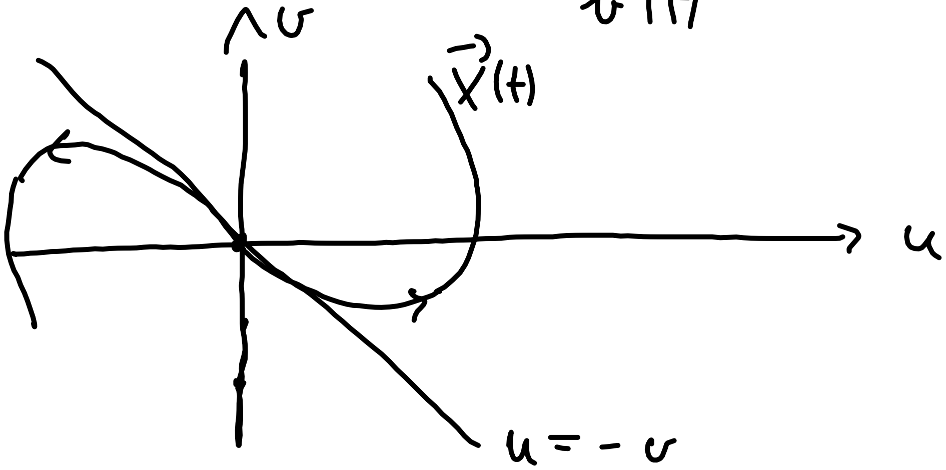
<u>General solution of $\vec{X}' = P\vec{X}$:</u>

$$\begin{aligned} \vec{X}(t) &= c_1\vec{X}_1(t) + c_2\vec{X}_2(t) \\ &= e^{4t}[c_1\vec{v}_1 + c_2(t\vec{v}_1 + \vec{v}_2)] \\ &= e^{4t}[(c_1 + tc_2)\vec{v}_1 + c_2\vec{v}_2] \\ &= e^{4t}\begin{bmatrix} -3c_1 - 3c_2 t \\ 3c_1 + (3t+1)c_2 \end{bmatrix} = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix} \end{aligned}$$

If $c_2 = 0$, then $\vec{X}(t) = \begin{bmatrix} -3c_1 \\ 3c_1 \end{bmatrix} e^{4t}$,

is a solution, that is the straight line $u = -v$ is a solution

As $t \to \pm\infty$, $\frac{\dot{a}(t)}{\dot{b}(t)} \longrightarrow -1$



$$\frac{\dot{a}(t)}{\dot{b}(t)} = \frac{e^{4t}(4(-3c_1-3c_2t)-3c_2) \cdot \frac{1}{t}}{e^{4t}(4(3c_1+(3t+1)c_2)+3c_2) \cdot \frac{1}{t}}$$

$$= \frac{\left(4\left(-\frac{3c_1}{t} - 3c_2\right) - 3\frac{c_2}{t}\right)}{\left(4\left(\frac{3c_1}{t} + \left(3+\frac{1}{t}\right)c_2\right) + \frac{3c_2}{t}\right)}$$

$$\xrightarrow[t \to \pm\infty]{} \frac{-3c_2}{3c_2} = \underline{-1}$$

Generalized eigenvectors, general case.

$\vec{v}_2$ above : called a generalized eigenvector for λ

Def. Suppose λ is an eigenvalue of P ($n\times n$-matrix). Then a rank r generalized eigenvector for λ is a vector $\vec{v}$ such that

$$(P-\lambda I)^r \vec{v} = \vec{0} \quad \text{but} \quad (P-\lambda I)^{r-1}\vec{v} \neq \vec{0}$$

Remark. If $r = 1$, then $\vec{v}$ is an eigenvector for λ.

The vector $\vec{v}_2$ of last example is a rank 2 generalized eigenvector.

Def. A k-chain (length k chain) of generalized eigenvectors based on an eigenvector $\vec{v}_1$, is a set

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$$

of generalized eigenvectors for λ such that

$$\left.\begin{aligned} (P-\lambda I)\vec{v}_k &= \vec{v}_{k-1} \\ (P-\lambda I)\vec{v}_{k-1} &= \vec{v}_{k-2} \\ &\cdots \\ (P-\lambda I)\vec{v}_2 \left[= (P-\lambda I)^{k-1}\vec{v}_k\right] &= \vec{v}_1 \end{aligned}\right\} (*)$$

<u>Notice</u> that

$$(P-\lambda I)^k \vec{v}_k = (P-\lambda I)\vec{v}_1 = \vec{0}$$

since $\vec{v}_1$ is an eigenvector for λ.

<u>$k=3$:</u> Assume that λ is a multiple eigenvalue for P. If $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a 3-chain of generalized eigenvectors for λ, then:

<u>Claim.</u> Three linearly independent solutions of $\vec{X}' = P\vec{X}$ are given by

$$(*)\begin{cases} \vec{X}_1(t) = e^{\lambda t}\vec{v}_1 \\ \vec{X}_2(t) = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2) \\ \vec{X}_3(t) = e^{\lambda t}\left(\frac{1}{2}t^2\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right) \end{cases}$$

Proof. We already know $\vec{X}_1$ is a solution, as $\vec{v}_1$ is an eigenvector.

From $(*)$ above:

$$(P-\lambda I)\vec{v}_3 = \vec{v}_2 \Rightarrow P\vec{v}_3 = \vec{v}_2 + \lambda\vec{v}_3$$

$$(P-\lambda I)\vec{v}_2 = \vec{v}_1 \Rightarrow P\vec{v}_2 = \vec{v}_1 + \lambda\vec{v}_2$$

Hence

$$P\vec{X}_3(t) = e^{\lambda t}\left[\lambda\vec{v}_1 + (\lambda t\vec{v}_1 + P\vec{v}_2) + \left(\frac{1}{2}t^2\lambda\vec{v}_1 + tP\vec{v}_2 + P\vec{v}_3\right)\right]$$

$$= e^{\lambda t}\left[\lambda\left(1+t+\frac{1}{2}t^2\right)\vec{v}_1 + \vec{v}_1 + \lambda\vec{v}_2 + t(\vec{v}_1 + \lambda\vec{v}_2) + \vec{v}_2 + \lambda\vec{v}_3\right]$$

$$\frac{d}{dt}\vec{X}_3(t) = e^{\lambda t}\left[\lambda\left(\frac{1}{2}t^2\vec{v}_1 + t\vec{v}_2 + \vec{v}_3\right) + (t\vec{v}_1 + \vec{v}_2)\right]$$

It follows that (after some calculations)

$$P\vec{X}_3(t) = \frac{d}{dt}\vec{X}_3(t)$$

Similarly $P\vec{X}_2 = \vec{X}_2{}'$.

$\{\vec{X}_1, \vec{X}_2, \vec{X}_3\}$ are lin. independent:

Consider

$$\vec{F}(t) = \alpha \vec{x}_1(t) + \beta \vec{x}_2(t) + \gamma \vec{x}_3(t), \; t \in I$$

Suppose $\alpha \vec{x}_1 + \beta \vec{x}_2 + \gamma \vec{x}_3 = \vec{0}$ - function.

Then (cancelling with $e^{\lambda t}$)

$$\vec{0} = \vec{F}(t) = \alpha \vec{v}_1 + (\beta \vec{v}_1 + \gamma \vec{v}_2) t + \tfrac{1}{2} \gamma \vec{v}_1 t^2$$

so all coefficients of the (vector) polynomial must be $\vec{0}$. Hence

$\frac{1}{2} \gamma \vec{v}_1 = \vec{0}$, so $\underline{\gamma = 0}$ $(\vec{v}_1 \neq 0)$

so $\beta \vec{v}_1 + \gamma \vec{v}_2 = \beta \vec{v}_1 = \vec{0}$, $\underline{\beta = 0}$

Finally $\alpha \vec{v}_1 = \vec{0}$, so $\underline{\alpha = 0}$

Hence $\underline{\vec{x}_1, \vec{x}_2, \vec{x}_3}$ are lin. independent.

This method extends to rank r case. Suppose

$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is a k-chain

of generalized eigenvectors for λ.

A set of k lin. indep. solutions of $\vec{x}' = P\vec{x}$

is given by

$$\vec{x}_1(t) = e^{\lambda t}\,\vec{v}_1$$

$$\vec{x}_2(t) = e^{\lambda t}(t\vec{v}_1 + \vec{v}_2)$$

$$\vdots$$

$$\vec{x}_k(t) = e^{\lambda t}\left(\frac{t^{k-1}}{(k-1)!}\vec{v}_1 + \dots + \frac{t^2}{2!}\vec{v}_{k-2} + t\,\vec{v}_{k-1} + \vec{v}_k\right)$$
