19. April, 2016. Lecture 12

Recall:
If $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are $n$ linearly independent solutions of

(*) $\vec{x}\,' = A\vec{x}$ (where $A$ is an $n\times n$ matrix)

Then the matrix, called a fundamental matrix for (*),

$X(t) = \Phi(t) = [\vec{x}_1(t)\ \vec{x}_2(t) \dots \vec{x}_n(t)]$

satisfies the equation

(**) $X' = AX$,

that is, $[\vec{x}_1{}'\ \vec{x}_2{}' \dots \vec{x}_n{}'] = [A\vec{x}_1\ A\vec{x}_2 \dots A\vec{x}_n]$

Thm 1: $\vec{x}(t) = \Phi(t)\Phi(0)^{-1}\vec{x}_0$, $t\in I$ ($I$ an open interval) is the unique solution of the initial value problem

$\vec{x}\,' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$,

where $\Phi(t)$ is any fundamental matrix of the system, $t\in I$.

Thm. 2 The unique sol. of
$\vec{x}\,' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$
is $\vec{x}(t) = e^{tA}\vec{x}_0$

Here $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!}A^n$ (Euclidean norm on $\mathbb{R}^{n^2}$) (We can regard $A$ as an element of $\mathbb{R}^{n^2}$)

$\Phi(t) = e^{tA}$ is a fundamental matrix with $\Phi(0) = I$.

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Suppose $\vec{u}$ is a generalized eigenvector of rank $r\geq 1$ of an $n\times n$ matrix $A$. Then, if $\lambda$ is the corresponding eigenvalue of $A$,

(*) $(A-\lambda I)^r\vec{u} = \vec{0}$, $(A-\lambda I)^{r-1}\vec{u} \neq \vec{0}$

Let us show how to calculate a solution of

$\vec{x}\,' = A\vec{x}$

corresponding to $\vec{u}$:

$$\vec{x}(t) = e^{tA}\vec{u} = e^{t(\lambda I + A - \lambda I)}\vec{u} = e^{(t\lambda I) + t(A-\lambda I)}\vec{u}$$
$$= e^{\lambda t} I \, e^{t(A-\lambda I)}\vec{u}$$

(since $e^{\lambda t I} = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} I^n = \left(\sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!}\right) I = e^{\lambda t} I$ which commutes with all matrices, hence with $t(A-\lambda I)$)

$$\vec{x}(t) = e^{\lambda t} I \left( \sum_{n=0}^{\infty} \frac{t^n}{n!}(A-\lambda I)^n \vec{u} \right) \qquad (**)$$
$$= e^{\lambda t}\left( \sum_{n=0}^{r-1} \frac{t^n}{n!}(A-\lambda I)^n \vec{u} \right)$$

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Using $(**)$ we can find a fundam. matrix
$\Phi(t) = [\vec{x}_1(t) \dots \vec{x}_n(t)]$
of the system, by calculating $n$ linearly indep. generalized eigenvectors $\vec{u}_1, \vec{u}_2, \dots \vec{u}_n$ (using $(**)$)

## Nonhomogeneous systems and Variation of the Parameters

We wish to find a particular solution $\vec{x}_p$ of the nonhomogeneous linear system:

(1) $\vec{x}\,' = P(t)\vec{x} + \vec{f}(t)$

when we know a general solution $\vec{x}_c(t)$ of the homogeneous

(2) $\vec{x}\,' = P(t)\vec{x}$

Then (3) $\vec{x}_c(t) = \Phi(t)\vec{c} = c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t)$

where $\Phi(t)$ is any fundam. matrix.

Idea: Replace the vector $\vec{c}$ in (3) by a vector function $\vec{u}(t)$. Then try ~~to~~ choose $\vec{u}(t)$ such that $\Phi(t)\vec{u}(t)$ becomes a particular sol. of (1).

Thus

we seek a solution

$\vec{x}_p(t) = \Phi(t)\vec{u}(t)$

of (1). Then

$\vec{x}_p\,'(t) = \Phi'(t)\vec{u}(t) + \Phi(t)\vec{u}\,'(t)$

Thus (1) becomes

$$\cancel{\Phi'(t)\vec{u}(t)} + \Phi(t)\vec{u}\,'(t) = \cancel{\overbrace{P(t)\Phi(t)}^{\Phi'(t)}\vec{u}(t)} + \vec{f}(t)$$

or $\Phi(t)\vec{u}\,'(t) = \vec{f}(t)$ $\quad | \, \Phi(t)^{-1}$

or $\vec{u}\,'(t) = \Phi(t)^{-1}\vec{f}(t)$

or $\vec{u}(t) = \int \Phi(t)^{-1}\vec{f}(t)\,dt$ (integrate each coordinate of the vector)

(we can take the integration constant equal to zero)

Theorem

$$\boxed{\vec{x}_p(t) = \Phi(t)\vec{u}(t) = \Phi(t)\int \Phi(t)^{-1}\vec{f}(t)\,dt}$$

is a particular solution of

$\vec{x}\,'(t) = P(t)\vec{x}(t) + \vec{f}(t)$ (defined on some open interval $I$)

A general solution of (1) is therefore

$$\boxed{\vec{x}(t) = \Phi(t)\vec{c} + \Phi(t)\int \Phi(t)^{-1}\vec{f}(t)\,dt}$$

($\Phi(t)$ is any fundamental matrix.)

The solution

$\vec{x}_c(t) = \Phi(t)\Phi(a)^{-1}\vec{x}_a$

of $\vec{x}\,' = P(t)\vec{x}$, $\vec{x}(a) = \vec{x}_a$

then yields

$$\boxed{\vec{x}(t) = \Phi(t)\Phi(a)^{-1}\vec{x}_a + \Phi(t)\int_a^t \Phi(s)^{-1}\vec{f}(s)\,ds}$$

the solution of

$\vec{x}\,' = P(t)\vec{x} + \vec{f}(t)$

that satisfies $\vec{x}(a) = \vec{x}_a$

Example

$$\vec{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \vec{x} - \begin{bmatrix} 15 \\ 4 \end{bmatrix} t e^{-2t}, \quad \vec{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

The associated homogeneous eq. $\vec{x}' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \vec{x}$

has a general solution (with eigenvalues $\lambda = -2, 5$)

$$\vec{x}_c(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t} = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

The the corresp. fundam. matrix is

$$\Phi(t) = \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}, \quad \Phi(0)^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

Hence

$$e^{tA} = \Phi(t)\Phi(0)^{-1} = \frac{1}{7} \begin{bmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} e^{-2t} + 6e^{5t} & -2e^{-2t} + 2e^{5t} \\ -3e^{-2t} + 3e^{5t} & 6e^{-2t} + e^{5t} \end{bmatrix}$$

Then

$$\vec{x}_p(t) = e^{tA} \int_0^t e^{-sA} \left( - \begin{bmatrix} 15 \\ 4 \end{bmatrix} s e^{-2s} \right) ds$$

satisfies $\vec{x}_p(0) = \vec{0}$

Furthermore, we must solve (for $c_1, c_2$)

$$\vec{x}_c(0) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

or

$$\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}, \qquad 1 \cdot \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^{-1}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \underline{\frac{1}{7} \begin{bmatrix} 1 \\ 24 \end{bmatrix}}, \quad c_1 = \frac{1}{7}, \; c_2 = \frac{24}{7}$$

so $\vec{x}(t) = \vec{x}_c(t) + \vec{x}_p(t)$ solves the initial value problem.

After calculations we find

$$\vec{x}(t) = \frac{1}{14} \begin{bmatrix} (6 + 28t - 7t^2)e^{-2t} & 92e^{5t} \\ (-4 + 14t + 21t^2)e^{-2t} & 46e^{5t} \end{bmatrix}$$

(see the book, E?).

Example Consider

$$(*) \quad \vec{x}\,' = A\vec{x}, \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

We shall find the general solution, and then $e^{tA}$.

Eigenvalues : 1, 2, 2.

Eigenvectors $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ :

$\lambda = 1$: $A - I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Here $c = 0$, $b = 0$, $a$ arbitrary. $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ spans the eigenspace.

$\lambda = 2$: $A - 2I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $-a + c = 0$, $a = c$, $b$ arbitrary. Basis for eigenspace: $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \vec{v}_3$.

A general sol. is

$$\vec{x}(t) = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 e^{t} + c_3 e^{2t} \\ c_2 e^{2t} \\ c_3 e^{2t} \end{bmatrix} \qquad (c_1, c_2, c_3 \text{ arbitrary numbers})$$

A fundamental matrix :

$$\Phi(t) = \begin{bmatrix} e^{t} & 0 & e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}, \qquad \Phi(0) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Phi(0)^{-1} = \frac{1}{1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{tA} = \Phi(t)\Phi(0)^{-1} = \begin{bmatrix} e^{t} & 0 & e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} e^{t} & 0 & e^{2t} - e^{t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix}}$$

Could also have found

$$e^{tA} = I + tA + \frac{1}{2!}t^2A^2 + \cdots$$

but :

$$A = B + D = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

where $BD \neq DB$, so calculations will be more laborous.