MAT 2440, April 26. 2016. Lecture

Will study problems like

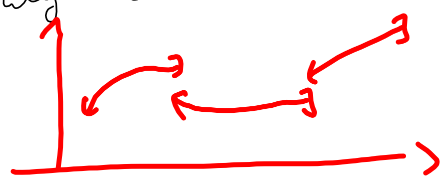
$$(1)\quad \begin{cases} \max\limits_{u} \int_{t_0}^{t_1} F(t, x(t), u(t))\, dt, & \underbrace{\dot{x}(t) = g(t, x(t), u(t))}_{\text{(the equation of state)}} \\ x(t_0) = x_0, \quad x(t_1) \text{ free}, \\ u(t) \in \mathbb{R}, \quad u \text{ and } \dot{x} \text{ piecewise continuous} \end{cases}$$

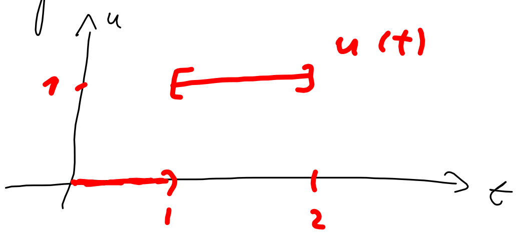
Usually: F, F_x, F_t, g, g_x, g_t are continuous.

Pairs (x, u) that satisfy the endpoint conditions of (1) and, in addition, the "state equation" $\dot{x}(t) = g(t, x(t), u(t))$, are called admissible (for the problem).

We are seeking optimal pairs (x^*, u^*), that is pairs that are admissible and that solves the max. problem.

That u and $\dot{x}$ are piecewise continuous on the (bounded) interval $[t_0, t_1]$, means they have only a finite number of jump discontinuities.





Useful to consider the function H,

$$H(t, x, u, p) = F(t, x, u) + p \cdot g(t, x, u),$$

called the Hamiltonian of the problem (the Hamilton function).

The Maximum Principle (of Pontryagin) gives necessary conditions for (x, u) to solve a given control problem.

The Maximum Principle (1) (Necessary conditions)

Suppose that (x^*, u^*) is an optimal pair for the problem (1). Then there exists a continuous function $p = p(t)$ such that, for all t in $[t_0, t_1]$, the following hold:

(a) $u^*(t)$ maximizes $H(t, x^*(t), u, p(t))$, $u \in \mathbb{R}$, that is $H(t, x^*(t), u, p(t)) \leq H(t, x^*(t), u^*(t), p(t))$ for all $u \in \mathbb{R}$.

(b) The function p (called the <u>adjoint function</u>) satisfies the differential equation

$$\dot{p}(t) = -\frac{\partial H}{\partial x}(t, x^*(t), u^*(t), p(t)) \quad \left(\text{written: } -\frac{\partial H^*}{\partial x}\right)$$

except at the (jump) discontinuities of u^*.

(c) The condition

(T) $p(t_1) = 0$ (transversality)

Remark. For our problems in the calculus of variations where $\boxed{\dot{x} = u}$, the Hamiltonian will be simple:

$$H(t, x, u, p) = F(t, x, u) + pu$$

Hence the Max. Principle tells us that for any optimal x^*,

(i) $\underline{\dot{p}(t)} = -\frac{\partial H^*}{\partial x} = \underline{-\frac{\partial F^*}{\partial x}}$

Since $u \in \mathbb{R}$ (which has no endpoints), the necessary condition for a max of H is that

$$0 = \frac{\partial H^*}{\partial u} = +\frac{\partial F^*}{\partial u} + p(t)$$

Hence

(ii) $p(t) = \underline{-\frac{\partial F^*}{\partial u} = -\frac{\partial F^*}{\partial \dot{x}}}$

By (ii) and (i) we obtain

$$\dot{p}(t) \overset{(ii)}{=} -\frac{d}{dt}\left(\frac{\partial F^*}{\partial \dot{x}}\right) \overset{(i)}{=} -\frac{\partial F^*}{\partial x}$$

That is, the Euler eq. must hold:

(E) $$\frac{\partial F^*}{\partial x} - \frac{d}{dt}\left(\frac{\partial F^*}{\partial \dot{x}}\right) = 0$$

Furthermore, as $p(t) = -\frac{\partial F^*}{\partial \dot{x}}$, (ii) and (T) yield:

$p(t_1) \overset{(T)}{=} 0$, i.e., $\underline{\left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t=t_1} = 0}$ (the "old" transversality cond)

<u>Obligatory problems:</u>

New chance May 10, 2:15 p.m.

(pick up solution at 7th floor Math. building, special Oblig. shelves)

Mangasarian's Theorem (Sufficiency condition)

Under the conditions of the Maximum Principle, suppose that the function h_t,

$$h_t : (x, u) \longmapsto H(t, x, u, p(t))$$

is concave for each t in $[t_0, t_1]$. Then each admissible pair (x, u) that satisfies (a), (b), and (c) in the Maximum Principle is optimal for the problem.

Example 1. $\max\limits_{u} \int_0^T [1 - t x(t) - u(t)^2]\, dt$, $\dot{x}(t) = u(t)$

$x(0) = x_0$, $x(T)$ free; where $x_0 > 0$, $T > 0$ are constants.

Solution: The Hamiltonian,

$$H(t, x, u, p) = 1 - tx - u^2 + pu$$

In order that $u = u^*(t)$ shall maximize $H(t, x, u, p)$, it is necessary that

$$\frac{\partial H}{\partial u}(t, x^*(t), u, p(t)) = 0$$

That is, $\quad -2u + p(t) = 0$

or $\quad u = \frac{1}{2} p(t)$

Since $\quad \frac{\partial^2 H}{\partial u^2} = -2 < 0$,

this will give a maximum. Hence

$$u^*(t) = \frac{1}{2} p(t), \quad t \in [0, T]$$

Moreover, $\frac{\partial H}{\partial x} = -t = -\dot{p}(t)$

Hence $\quad \dot{p}(t) = t, \quad t \in [0, T]$

$$p(t) = \frac{1}{2} t^2 + A$$

Furthermore, $0 = p(T) = \frac{1}{2} T^2 + A$, $\underline{A = -\frac{1}{2} T^2}$

$$p(t) = \frac{1}{2} t^2 - \frac{1}{2} T^2 = \underline{\frac{1}{2}(t^2 - T^2)}$$

$$u^*(t) = \frac{1}{2} p(t) = \underline{\frac{1}{4}(t^2 - T^2)}$$

Now the "State equation" yields

$$\dot{x}^*(t) = u^*(t) = \frac{1}{2}p(t) = \frac{1}{4}(t^2 - T^2)$$

$$x^*(t) = \frac{1}{4}\left(\frac{1}{3}t^3 - T^2 t\right) + B$$

$$= \frac{1}{12}t^3 - \frac{1}{4}T^2 t + x_0$$

Since $H(t, x, u, p(t)) = 1 - tx - u^2 + pu$

is the sum of the 2 concave functions

$(x, u) \mapsto pu - tx$ (linear of (x,u), hence concave)

and $(x, u) \mapsto 1 - u^2$ (concave of u, constant in x, hence concave),

H must be concave in (x, u).

Alternative: 2nd derivative test:

$$\frac{\partial H}{\partial x} = -t, \quad \frac{\partial^2 H}{\partial x^2} = 0 \le 0$$

$$\frac{\partial H}{\partial u} = -2u + p, \quad \frac{\partial^2 H}{\partial u^2} = -2 \le 0$$

$$\frac{\partial^2 H}{\partial u \partial x} = 0$$ hence

$$\Delta = \begin{vmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial x \partial u} & \frac{\partial^2 H}{\partial u^2} \end{vmatrix} = 0 \cdot (-2) - 0^2 = 0 \ge 0 \text{ on } [0, T].$$

Hence Mangasarian's Theorem yields that (x^*, u^*) maximizes the integral.

Example 2. $\max_u \int_0^T (x-u^2)\,dt$, $\dot{x} = x+u$, $x(0)=0$, $x(T)$ free, $u \in \mathbb{R}$.

Solution: $H(t,x,u,p) = x-u^2 + p(x+u)$

$$= \underbrace{(-u^2+pu)}_{\text{concave of } u} + \underbrace{(1+p)x}_{\text{linear in } x}$$

H is concave in (x,u), as a sum of 2 concave functions. We will maximize H wrt. u:

$$\frac{\partial H}{\partial u} = -2u + p + 0 = 0, \quad u = \frac{1}{2}p$$

This gives a max. since $\frac{\partial^2 H}{\partial u^2} = -2 < 0$.

Hence $\dot{x} = x+u = x + \frac{1}{2}p$

and $\dot{p} = -\frac{\partial H^*}{\partial x} = -(1+p)$; $\dot{p}+p = -1$.

$\dot{p}+p=0$ has the general solution: $p_c(t) = Ae^{-t}$

particular solution: $p_p(t) = -1$.

Hence $p(t) = Ae^{-t} - 1$

Now (T): $p(T) = 0$, $Ae^{-T} = 1$, $\underline{A = e^T}$

$$\underline{p(t) = e^{T-t} - 1}$$

Hence $\dot{x} - x = \frac{1}{2}p = \frac{1}{2}(e^{T-t}-1) \quad / \cdot e^{-t} = e^{\int -1\,dt}$

$$\frac{d}{dt}(e^t x) = \frac{1}{2}(e^{T-t}-1)e^{-t} = \frac{1}{2}(e^{T-2t} - e^{-t})$$

$$e^t x = \frac{1}{2}\left[-\frac{1}{2}e^{T-2t} + e^{-t}\right] + D = -\frac{1}{4}e^{T-2t} + \frac{1}{2}e^{-t} + D$$

$$x(t) = -\frac{1}{4}e^{T-t} + \frac{1}{2} + De^t$$

$$x(0) = 0 = -\frac{1}{4}\left(e^T + \frac{1}{2} + D\right), \quad D = \frac{1}{2}\left(-1 + \frac{1}{2}e^T\right) = -\frac{1}{2} + \frac{1}{4}e^T$$

$$x^*(t) = -\frac{1}{4}e^{T-t} + \frac{1}{4}e^{T+t} - \frac{1}{2}e^t + \frac{1}{2}$$

$$= \underline{\frac{1}{4}\left(e^{T+t} - e^{T-t}\right) + \frac{1}{2}(1-e^t)}$$

x^* solves the problem by Mangasarian's Thm.