The Compulsory problem sets are now available at 7th floor of the Math. Building (the "Ollig Shelves"). (Contact me if your should be missing)

A more general Maximum Principle.

We shall next study control problems of the following type:

$$(*)\begin{cases} \max \int_{t_0}^{t_1} f(t, x(t), u(t))\, dt \\ \dot{x}(t) = g(t, x(t), u(t)), \quad x_0 = x(t_0) \text{ given}, \quad u(t) \in \mathcal{U}, \ (t \in [t_0, t_1]), \\ \text{where } \mathcal{U} \text{ is a given interval in } \mathbb{R}. \end{cases}$$

We assume that f and g are C^1-functions (real valued).

Moreover, we suppose that $x(t_1)$ satisfies exactly one of the following 3 conditions:

(i) $x(t_1) = x_1$, x_1 a given real value (fixed)

(ii) $x(t_1) \geq x_1$, — " —

(iii) $x(t_1)$ is free (no condition on $x(t_1)$)

In order that the Maximum Principle shall hold, it now becomes necessary to consider the following two possibilities for the Hamiltonian H:

(A) $H(t, x, u, p) = f(t, x, u) + p\, g(t, x, u)$ (Normal problems)

(B) $H(t, x, u, p) = p\, g(t, x, u)$ (Non-normal or degenerate problems)

(A) and (B) may be written in one formula:

$$H(t, x, u, p) = p_0 f(t, x, u) + p\, g(t, x, u)$$

where $p_0 = 1$ or $p_0 = 0$

Remark. If $x(t_1)$ is free (case (iii) above), then the problem will always be normal.

More generally, the problem is always normal if $p(t) = 0$ for some $t \in [t_0, t_1]$ (p is the adjoint function of the problem, given in the Maximum Principle, see below).

The Maximum Principle II (general version)

Suppose that (x^*, u^*) is an optimal pair for the problem $(*)$ with exactly one of the terminal conditions (i), (ii), (iii) satisfied. Then there is a continuous, piecewise differentiable function p (called the adjoint function of the problem) such that $p(t_1)$ satisfies exactly one of the following transversality conditions:

(i') no condition on $p(t_1)$ if (i) $x(t_1) = x_1$

(ii') $p(t_1) \geq 0$ if (ii) $x(t_1) \geq x_1$ ($p(t_1)=0$ if $x(t_1) > x_1$)

(iii') $p(t_1) = 0$ if (iii) $x(t_1)$ is free

In addition, for each $t \in [t_0, t_1]$, we must have

(a) $u = u^*(t)$ maximizes the function h_t, where $h_t(u) = H(t, x^*(t), u, p(t))$, for $u \in \mathcal{U}$ ($\in$ the control region)

(b) $\dfrac{\partial H^*}{\partial x} = -\dot{p}(t)$ (where $\dfrac{\partial H^*}{\partial x} = \dfrac{\partial H}{\partial x}(t, x^*(t), u^*(t), p(t))$)

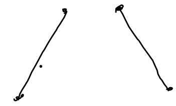
except at the discontinuities of u^*.

[(c) If $p(t_1) = 0$ for some $t \in [t_0, t_1]$, then the problem is normal]

As before we also have the following sufficient condition for a maximum:

Mangasarian's Theorem

Suppose that (x^*, u^*) is an admissible pair for the problem $(*)$ that satisfies all the conditions of the Maximum Principle. If for each $t \in [t_0, t_1]$, the function $(x, u) \longmapsto H(t, x, u, p(t))$ is concave, then (x^*, u^*) is an optimal pair for the problem (that is, (x^*, u^*) maximizes the integral in $(*)$)



Example 3. Solve (the normal problem)

$$\max_u \int_0^1 x(t)\,dt\ , \quad \dot{x} = x+u\ , \quad x(0)=0\ ,\ x(1)\geq 1$$
$$u(t) = [-1,1] = \mathcal{U}$$

Solution. Since we assume that the problem is normal,

$$H(t,x,u,p) = x + p(x+u) = (1+p)x + pu\ ,$$

which is linear (of first degree) in (x,u), hence

$$(x,u) \longmapsto H(t,x,u,p(t))$$

is both concave and convex. Therefore, Mangasarian's Thm applies. Here we must have:

(iii') $p(1)\geq 0$ $[p(1)=0$ if $x(1)>1]$

H is maximized as a function of u at one of the endpoints $u=1$ or $u=-1$ (since u is linear).

Hence

$$u^*(t) = \begin{cases} 1\ , & \text{if } p(t)>0 \\ -1\ , & \text{if } p(t)<0 \\ \text{can be chosen arbitrarily in } [-1,1], \text{ as } H \text{ is constant in } u \text{ if } p(t)=0 \\ \text{We choose } u^*(t)=1 \text{ in this case} \end{cases}$$

Furthermore, for $x=x^*$, $u=u^*$, we must have

$$\frac{\partial H}{\partial x} = 1+p = -\dot{p} \qquad \text{(from (b) of the Max. Principle)}$$

Hence $\dot{p}+p=-1$

Thus $p(t) = Ae^{-t}-1$,

where $p(1) = Ae^{-1}-1 \geq 0$ (by the max. principle)

Hence $Ae^{-1}\geq 1$, $\underline{A\geq e}$

Then $p(t) = Ae^{-t}-1 \boxed{\geq} e\,e^{-t}-1 = \underbrace{e^{1-t}-1} := h(t)$,

where $\dot{h}(t) = -e^{1-t}<0$ and $h(1) = e^0-1 = \underline{0}$

Moreover, $h(0) = e-1>0$, $t\in[0,1]$, so that

$$h(t)\in[0,\ e-1].$$

In particular, $p(t)\geq h(t)>0$, $t\in[0,1)$, $p(1)\geq 0$.

Hence H as a function of u is increasing, so

$$u = u^*(1) = 1\ , \quad t\in[0,1].$$

Hence

$$\dot{x}^*-x^* = 1\ ,$$
$$x^*(t) = Be^t-1$$

$x^*(0)=0 \Rightarrow B=1 \Rightarrow x^*(t) = \underline{e^t-1}$

Hence $x^*(1) = e-1>1$, so $p(1) = 0$, so $A=1$.

$\underline{x^*(t) = e^t-1\ ,\quad u^*(t) = 1\ ,\quad t\in[0,1],}$

is then an optimal pair in light of Mangasarian's Thm.

Example 4. Solve the (normal) problem:

$$\max \int_0^1 (2x - x^2)\,dt, \quad \dot{x} = u, \quad x(0) = x(1) = 0, \quad -1 \le u(t) \le 1$$

Solution.

Here $H(t,x,u,p) = 2x - x^2 + pu$ is concave in (x,u): (By the "2nd derivative test")

$$\frac{\partial^2 H}{\partial x^2} = -2 \le 0, \quad \frac{\partial^2 H}{\partial u^2} = 0 \le 0, \quad \frac{\partial^2 H}{\partial x \partial u} = 0,$$

$$\Delta_H = \begin{vmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial x \partial u} & \frac{\partial^2 H}{\partial u^2} \end{vmatrix} = (-2)\cdot 0 - 0^2 = 0 \ge \underline{0}$$

Mangasarian's Thm. therefore applies.

H is linear in u so the maximum of H occurs at an endpoint $u = -1$, $u = 1$. Hence, for any optimal pair (x^*, u^*) we have

$$u^*(t) = \begin{cases} 1, & \text{if } p(t) \ge 0 \\ -1, & \text{if } p(t) < 0 \end{cases} \quad (\text{choosing } u^* = 1 \text{ if } p(t) = 0)$$

Furthermore,

$$\dot{p}(t) = -\frac{\partial H^*}{\partial x} = -(2 - 2x^*(t)) = -2 + 2x^*(t)$$

Since $x^*(0) = 0$ and $\underline{\dot{x}^* = u^* \le 1}$, so

$$\int_0^t \dot{x}^*(s)\,ds = x^*(t) - x^*(0) = x^*(t) \le \int_0^t 1\,ds = t \le 1$$

so that $x^*(t) < 1$, $t \in [0,1)$.

Hence $\dot{p}(t) = 2(x^*(t) - 1) < 0$, $t \in [0,1)$

So p is strictly decreasing on $[0,1)$.

Next, if we consider the following possibilities:

(1) $\underline{p(0) \le 0}$, then $p(t) < 0$ on $(0,1]$, then $u^*(t) = -1$ on $(0,1]$,
hence $\dot{x}^* = -1$, so $x^*(t) = -t$ on $(0,1]$.
Impossible since $x^*(1) = 0$!

(2) $\underline{p(1) \ge 0}$, then $p(t) > 0$ on $[0,1)$ (p was strictly decreasing on $[0,1]$),
hence $u^* = \dot{x}^* = 1$, so $x^*(t) = t$, impossible as $x^*(1) = 0$.

Hence $p(0) > 0$ and $p(1) < 0$

Then since p is strictly decreasing on $[0,1)$, so there is a unique t_0 on $(0,1)$ such that $\underline{p(t_0) = 0}$.

Hence

$$\dot{x}^*(t) = u^*(t) = \begin{cases} 1, & t \in [0, t_0) \\ -1, & t \in (t_0, 1] \\ \text{undetermined} & \text{if } t = t_0 \end{cases}$$

As $x^*(0) = 0$, we have
$x^*(t) = t$, $t \in [0, t_0)$
and since $x^*(1) = 0$, we have
$x^*(t) = -t + 1$, $t \in (t_0, 1]$.

Since x^* is continuous at t_0, we have

$$t_0 = \lim_{t \to t_0^-} x^*(t) = \lim_{t \to t_0^+} x^*(t) = -t_0 + 1$$

Hence $\underline{t_0 = \frac{1}{2}}$

Then we know $\underline{u^*(t) \text{ and } x^*(t)}$

This is an optimal pair Mangasarian's Thm.