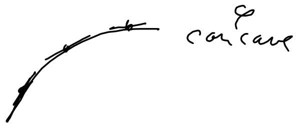
MAT 2440. May 3, 2016. Lecture 15

Before we prove Mangasarian's Thm., we will need a lemma:

Lemma Suppose φ is a $\boxed{\text{concave}}$ C^1-function on an interval I in $\mathbb{R}$. Then



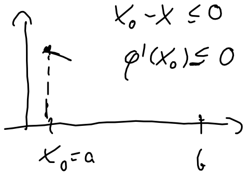
φ has a max point at $x_0 \in I$

$\Updownarrow$ (if and only if)

$\varphi'(x_0)\cdot(x_0 - x) \geq 0$, for all $x \in I$

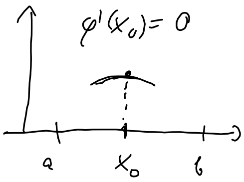
Proof. $\Downarrow$: We let $a < b$, a and b the endpoints of I. Suppose that x_0 is a max point of φ. Since φ' exists, there are exactly 3 possibilities

1) $x_0 = a$,
$x_0 - x \leq 0$
$\varphi'(x_0) \leq 0$



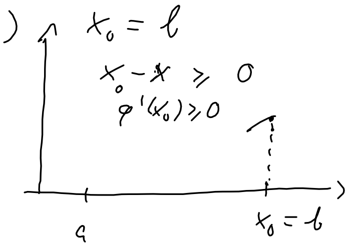
$\varphi'(x_0)\cdot(x_0 - x) \geq 0$

2) $a < x_0 < b$
$\varphi'(x_0) = 0$



$\varphi'(x_0)(x_0 - x) = 0 \geq 0$

3) $x_0 = b$
$x_0 - x \geq 0$
$\varphi'(x_0) \geq 0$



$\varphi'(x_0)(x_0 - x) \geq 0$

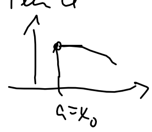
$\Uparrow$: Assume that $\varphi'(x_0)(x_0 - x) \geq 0$.
Again there are 3 possibilities:

1) $\underline{x_0 = a}$: $x_0 - x \leq 0$, hence $\varphi'(x_0) \leq 0$ (since $\varphi'(x_0)(x_0 - x) \geq 0$)

Since φ is concave and φ' exists, the tangent lie (over) above or on the graph of φ at each point t in I. Hence φ' decreases (can be proved analytically), as is "obvious" geometrically. Hence

$$\varphi'(x) \leq \varphi'(a) = \varphi'(x_0) \leq 0 \quad , \quad x \in I.$$

Hence φ is decreasing. Therefore, $x_0 = a$ is a max. point.



2) $a < x_0 < b$: (i) If $x \in I$ and $x_0 < x$, then $x_0 - x < 0$, hence $\varphi'(x_0) \le 0$.

(ii) $x \in I$ and $x < x_0$: Then $x_0 - x > 0$ and hence $\varphi'(x_0) \ge 0$ (since $\varphi'(x_0)(x_0 - x) \ge 0$)

Combining (i) and (ii), we have

$$\underline{\varphi'(x_0) = 0}$$

Since φ is concave, φ' is decreasing (as in 1)). Hence x_0 must be a max point.

3) $\underline{x_0 = b}$: $x_0 - x \ge 0$, hence $\varphi'(x_0) \ge 0$, for all x in I. Hence φ is increasing (argue as in 1)) Hence $x_0 = b$ is a max. point. ∎

<u>Remarks:</u> If I has only one endpoint, then the same proof applies. If I has no endpoint, then we are left with just case 2) above. ∎
Hence the lemma holds in all cases.

<u>Proof of Mangasarian's Thm.</u> Assume the problem is a <u>normal</u> problem, so "$H = f + pg$"

Suppose that (x^*, u^*) is an admissible pair for our control problem that satisfies all the conditions of the Maximum Principle, that is, the conditions (a), (b), and (c).

> $\max_u \int_{t_0}^{t_1} f(t, x(t), u(t))\,dt$
> $\dot{x}(t) = g(t, x(t), u(t))$
> $t \in [t_0, t_1]$, and
> $x(t_0) = x_0$, x_0 given,
> and exactly one of the terminal conditions (i), (ii), (iii) is satisfied for $x(t_1)$

Assume, in addition, that

$$h_t : (x, u) \longmapsto H(t, x, u, p(t))$$

is <u>concave</u> for each t in $[t_0, t_1]$.

Let (x, u) be any <u>admissible</u> pair for the problem. We must prove that

$$\Delta = \int_{t_0}^{t_1} f(t, x^*(t), u^*(t))\,dt - \int_{t_0}^{t_1} f(t, x(t), u(t))\,dt \ge 0$$

$\int_{t_0}^{t_1} (f^* - f)\,dt$

We introduce the simplified notation:

$f^* = f(t, x^*(t), u^*(t))$, $f = f(t, x(t), u(t))$

and similarly for g^*, g, H^*, H:

$[H^* = H(t, x^*(t), u^*(t), p(t))$ etc. ... $]$

Then $H^* = f^* + p g^*$, hence

$$f^* = H^* - p g^* = H^* - p \dot{x}^* \quad (\text{since } g^* = \dot{x}^*)$$

and $f = H - p\dot{x}$.

Therefore

$$\Delta = \int_{t_0}^{t_1} (f^* - f)\,dt = \int_{t_0}^{t_1} (H^* - p\dot{x}^* - H + p\dot{x})\,dt$$

$$= \int_{t_0}^{t_1} p(\dot{x} - \dot{x}^*)\,dt - \int_{t_0}^{t_1} (H - H^*)\,dt$$

<u>Arrow's Condition.</u>

If the Hamiltonian H fails to be concave in (x, u), we may sometimes use Arrow's condition.

Let $\hat{H}(t, x, p) = \max_{u \in \mathcal{U}} H(t, x, u, p)$

where we suppose that the maximum exists.

Then:

<u>Arrow's Sufficiency condition.</u>

Given the control problem (*) above. Suppose the Hamiltonian for the problem is

$$H(t, x, u, p) = f(t, x, u) + p\, g(t, x, u) \quad (\text{Normal Problem})$$

If the conditions (a), (b), (c) of the Max. Princ. are satisfied at (x^*, u^*), and $\hat{H}(t, x, p(t))$ is concave of x, for each $t \in [t_0, t_1]$, then (x^*, u^*) solves the problem.

<u>Example</u> (Arrow's Condition)

$$\max \int_1^2 (-x^2 + txu)\,dt, \quad \dot{x} = u, \quad x(1) = -\tfrac{1}{2},\ x(2) = 0$$
$$u(t) \in [0, 1] = \mathcal{U}, \quad t \in [1, 2].$$

Here, if we assume this is a normal problem,

$$H(t, x, u, p) = -x^2 + txu + pu$$
$$= -x^2 + (tx + p)u$$

which is linear in u, so attains its max. for $u = 1$ or $u = 0$. Hence

$$u^*(t) = \begin{cases} 1, & \text{if } tx + p > 0 \\ 0, & \text{if } tx + p < 0 \\ \text{undetermined} & \text{if } tx + p = 0 \end{cases}$$

The 2nd derivative test shows that H is not concave of (x, u) for any $t \in [1, 2]$

$$\left(\frac{\partial^2 H}{\partial x^2} = -2 < 0,\ \frac{\partial^2 H}{\partial u^2} = 0,\ \frac{\partial^2 H}{\partial x \partial u} = t,\ \Delta = (-2)\cdot 0 - t^2 = -t^2 < 0,\ t \in [1, 2] \right)$$

Here

$$\hat{H}(t, x, p(t)) = \begin{cases} -x^2, & \text{if } tx + p(t) \le 0 \\ -x^2 + tx + p(t), & \text{if } tx + p(t) > 0 \end{cases}$$

which is concave of x, for each $t \in [1, 2]$

so Arrow's Condition applies.

$$\Delta = \int_{t_0}^{t_1} p(\dot{x} - \dot{x}^*)\,dt - \int_{t_0}^{t_1} (\underbrace{H - H^*})\,dt$$

(Recall now the "Gradient inequality" for concave functions φ defined on an open convex set in $\mathbb{R}^n$ (SSS 4.6.1 / SHSS 2.4.1):

$$\underbrace{\varphi(\vec{x}) - \varphi(\vec{z}) \le \nabla\varphi(\vec{z}) \cdot (\vec{x} - \vec{z})},\ )$$

Thus, by the Gradient inequality:

$$\begin{aligned} H - H^* &\le \nabla H^* \cdot ((x,u) - (x^*,u^*)) \\ &= \underbrace{\frac{\partial H^*}{\partial x}(x - x^*)} + \frac{\partial H^*}{\partial u}(u - u^*) \\ &= -\dot{p}(x - x^*) + \underbrace{\frac{\partial H^*}{\partial u}(u - u^*)} \quad \text{(condition b) in Max Princ.)} \\ &\le -\dot{p}(x - x^*) + 0 \ \underline{\text{Lemma}} \end{aligned}$$

(except at the discontinuities of u^*)

Hence

$$\begin{aligned} \Delta &\ge \int_{t_0}^{t_1} \underbrace{p(\dot{x} - \dot{x}^*)}\,dt + \int_{t_0}^{t_1} \dot{p}(x - x^*)\,dt \\ &= \int_{t_0}^{t_1} \frac{d}{dt}\left[p(x - x^*)\right]dt \quad \text{(product rule)} \\ &= \Big|_{t_0}^{t_1} p(x - x^*) = p(t_1)(x(t_1) - x^*(t_1)) - p(t_0)\underbrace{\overbrace{(x(t_0) - x^*(t_0))}^{x_0 = x(t_0) = x^*(t_0)}}_{0} \\ &= p(t_1)(x(t_1) - x^*(t_1)) \end{aligned}$$

By the terminal conditions:

(i) $x(t_1) = x_1 = x^*(t_1)$: $\quad \Delta \ge 0$ for all $p(t_1)$

(ii) $x(t_1) \ge x_1$: $\underline{\text{If } x^*(t_1) = x_1}$, then $p(t_1) \ge 0$, hence $\Delta \ge 0$ since $x(t_1) \ge x_1 = x^*(t_1)$, so $\underline{p(t_1)\cdot(x(t_1) - x^*(t_1)) \ge 0}$

If $\underline{x^*(t_1) > x_1}$, then $p(t_1) = 0$, hence $p(t_1)(x(t_1) - x^*(t_1)) = 0$, hence

$$\underline{\Delta \ge 0}$$

(iii) $x(t_1)$ is free: Then $p(t_1) = 0$, hence

$$\underline{\Delta \ge 0}.$$

This completes the proof of Mangasarian's Thm.