Math 2440. Lecture 16. May 10. 2016

Linear and almost linear systems

We shall mainly study 2-dimensional autonomous systems:

(1) $\frac{dx}{dt} = f(x, y)$, $\frac{dy}{dt} = g(x, y)$ ($x = x(t)$, $y = y(t)$, $t \in I$, I an interval)

(autonomous: t does not occur explicitly in formulas for f and g)

Def. $(x_0, y_0) \in \mathbb{R}^2$ is a critical point of (1) if

$f(x_0, y_0) = 0 = g(x_0, y_0)$

(x_0, y_0) is an isolated critical point if there is a neighborhood $V_\delta(x_0, y_0) = \{(x, y) : \|(x, y) - (x_0, y_0)\| < \delta\}$ that contains no other critical points.

Here $\|(a, b)\| = \sqrt{a^2 + b^2}$, $(a, b) \in \mathbb{R}^2$.

Remark. If there is only finitely many critical points, $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, then the are all isolated. In fact, if

$$\delta = \min \{ \|(x_i, y_i) - (x_k, y_k)\| : \underset{i \neq k}{1 \le i, k \le n} \}$$

then $\delta > 0$. Then $V_\delta(x_i, y_i)$ contains no other critical point $(1 \le i \le k)$

We shall assume that f and g are C^1-functions in some neighborhood $V_\delta(x_0, y_0)$ $(\delta > 0)$ of any critical point (x_0, y_0), i.e., $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}$ are all continuous on $V_\delta(x_0, y_0)$.

By a change of coordinates

$$u = x - x_0 , \quad v = y - y_0 ,$$

(1) is transformed into an equivalent system which $(0, 0)$ is a critical point:

(2) $\frac{du}{dt} = f(u + x_0, v + y_0) = f_1(u, v)$, $\frac{dv}{dt} = g(u + x_0, v + y_0) = g_1(u, v)$

Linearization near a critical point.

f and g are C^1 near the critical point (x_0, y_0), so Taylor's formula in 2 variables yields:

$$f(x_0+u,\ y_0+v) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot u + \frac{\partial f}{\partial y}(x_0,y_0)\cdot v + r(u,v)$$

where $\displaystyle\lim_{(u,v)\to(0,0)} \frac{r(u,v)}{\sqrt{u^2+v^2}} = 0$

Similar equations hold for g.

Since (x_0, y_0) is a critical point, $f(x_0, y_0) = g(x_0,y_0) = 0$.

Hence (2) becomes

LINEAR

$$(2')\quad \frac{du}{dt} = \boxed{\frac{\partial f}{\partial x}(x_0,y_0)\, u + \frac{\partial f}{\partial y}(x_0,y_0)\cdot v} + r(u,v)$$

$$\frac{dv}{dt} = \boxed{\frac{\partial g}{\partial x}(x_0,y_0)\cdot u + \frac{\partial g}{\partial y}(x_0,y_0)\cdot v} + s(u,v)$$

where

$$(2'')\quad \frac{r(u,v)}{\sqrt{u^2+v^2}} \longrightarrow 0 \quad \text{and} \quad \frac{s(u,v)}{\sqrt{u^2+v^2}} \longrightarrow 0 \quad \text{as } (u,v)\to(0,0)$$

For <u>small</u> u and v, the terms $r(u,v)$ and $s(u,v)$ are <u>very</u> small (i.e., small when compared to u and v).

Omitting those small (usually nonlinear) terms we obtain a <u>linear</u> system

$$(3)\quad \left.\begin{aligned} \frac{du}{dt} &= \frac{\partial f}{\partial x}(x_0,y_0)\, u + \frac{\partial f}{\partial y}(x_0,y_0)\, v \\ \frac{dv}{dt} &= \frac{\partial g}{\partial x}(x_0,y_0)\, u + \frac{\partial g}{\partial x}(x_0,y_0)\, v \end{aligned}\right\} \quad \text{OR} \quad \begin{bmatrix}\dot u\\ \dot v\end{bmatrix} = J(x_0,y_0)\begin{bmatrix}u\\ v\end{bmatrix}$$

where $J(x_0,y_0) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{bmatrix}_{(x_0,y_0)}$

is the Jacobian matrix of (f,g) at (x_0, y_0)

Usually, we require that $\det J(x_0,y_0) \neq 0$.

Definition. We call (3) the linearization of (2) at (x_0, y_0) if (x_0, y_0) is an isolated critical point for (3) and $\det J(x_0, y_0) \neq 0$.

The system (2) (and (1)) is then called almost linear

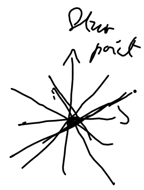
Remark. At a critical point (x_0, y_0), the constants $x_0(t) = C_1$, $y_0(t) = C_2$ are solutions of (1) (since $\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$), called equilibrium solutions

Suppose next that $J(x_0, y_0) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}_{(x_0, y_0)}$ has real eigenvalues λ_1, λ_2.

Def. Suppose $\lambda_1 \leq \lambda_2$. Then

1. (x_0, y_0) is a saddle point if $\lambda_1 < 0 < \lambda_2$
2. — " — a nodal sink if $\lambda_1 < \lambda_2 \leq 0$
3. — " — a nodal source if $0 \leq \lambda_1 < \lambda_2$
4. — " — a (proper or improper) node if $\lambda_1 = \lambda_2 = \lambda$
   "star point"
   It is a source if $\lambda > 0$, a sink if $\lambda < 0$

Star point



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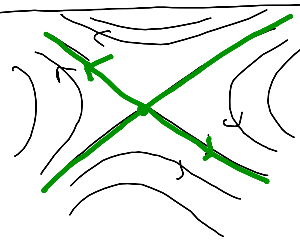
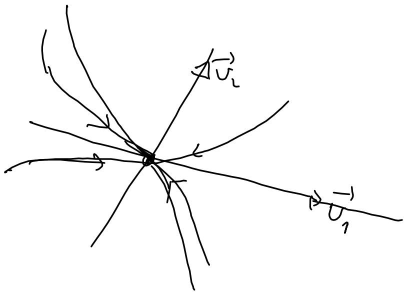
Typical sketches of trajectories (solution curves, paths):

1



Saddle point $\lambda_1 < 0 < \lambda_2$
trajectories resemble hyperbolas (of nonlinear system)

2

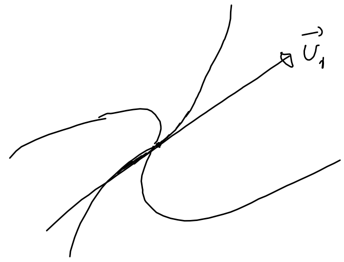


Nodal sink
$\lambda_1 < \lambda_2 \leq 0$

3.

Nodal source
(as nodal sink, but arrows points away from critical point)

4



Improper node
(sink or source)
trajectories are tangents to equilibrium solutions

Example.

(1) $\frac{dx}{dt} = (x+1)(3-x-y)$, $\frac{dy}{dt} = (y+1)(1-3x+y)$

The critical points are given by

$(x+1=0 \ \vee \ 3-x-y=0)$ and $(y+1=0 \ \vee \ 1-3x+y=0)$

Four critical points:

1 $x=-1, y=-1$ $\quad (-1,-1)$

2 $x=-1$ & $1-3x+y=0$, $\quad (-1,-4)$

3 $3-x-y=0$ & $y+1=0$, $\quad (4,-1)$

4. $3-x-y=0$ & $1-3x+y=0$ $\quad (1,2)$

(a) Consider first $(1,2)$: Let

$u = x-1$, $v = y-2$ $\quad (x = u+1, \ y = v+2)$

Then $3-x-y = 3-(u+1)-(v+2) = -(u+v)$

$1-3x+y = 1-3u-3+v+2 = -3u+v$

Hence

(2) $\begin{cases} \frac{du}{dt} = -(u+2)(u+v) = -u^2-2u-2v = -2u-2v-u^2 \\ \frac{dv}{dt} = (v+3)(-3u+v) = -9u+3v-3uv+v^2 \end{cases}$

Here $(0,0)$ is an isolated critical point (only finitely many).

The linearization of (2) around $(0,0)$ is

$$\left.\begin{aligned} \frac{du}{dt} &= -2u-2v \\ \frac{dv}{dt} &= -9u+3v \end{aligned}\right\} \quad \begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ -9 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = A \begin{bmatrix} u \\ v \end{bmatrix}$$

$A = J(0,0) = \begin{bmatrix} -2 & -2 \\ -9 & 3 \end{bmatrix}$, $\det A = -6-18 = -24 \neq \underline{0}$

Hence the system (2) is almost linear at $(0,0)$.

Eigenvalues

$$|A-\lambda I| = \begin{vmatrix} -2-\lambda & -2 \\ -9 & 3-\lambda \end{vmatrix} = (\lambda-3)(\lambda+2) - 18 = \lambda^2 - \lambda - 24 = 0$$

$$\lambda = \tfrac{1}{2}\left[1 \pm \sqrt{1+96}\right] = \tfrac{1}{2}\left[1 \pm \sqrt{97}\right] \ (= \lambda_1, \lambda_2)$$

2 real eigenvalues, $\lambda_1 < 0 < \lambda_2$: A saddle point

$\vec{v}_1, \vec{v}_2$ lin. indep. eigenvectors : $\begin{bmatrix} a \\ b \end{bmatrix}$

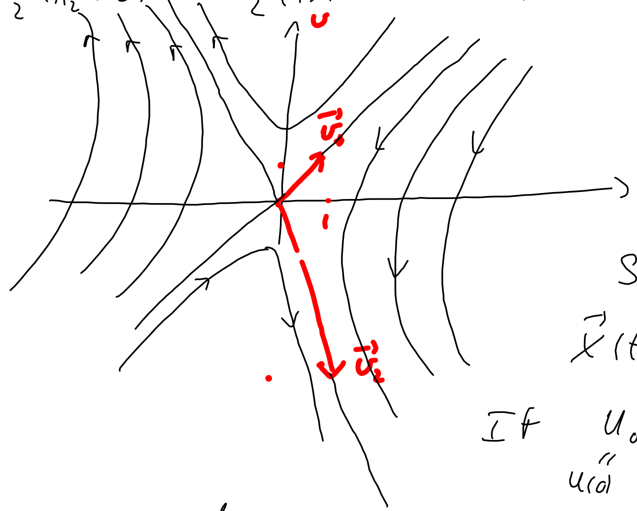
$\lambda_1 = \tfrac{1}{2}[1-\sqrt{97}] = -4.4\ldots$: $-(\lambda_1+2)a = 2b$

$a=1$ yields, $b = -\tfrac{1}{2}(\lambda_1+2) = -\tfrac{1}{2}(-2.4) = 1.2\ldots$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1.2\ldots \end{bmatrix}$

$\lambda_2 = \tfrac{1}{2}[1+\sqrt{97}] = 5.4\ldots$: $-(\lambda_2+2)a = 2b$. $a=1$ yields

$b = -\tfrac{1}{2}(\lambda_2+2) = -\tfrac{1}{2}(7.4\ldots) = -3.7\ldots$ $\vec{v}_2 = \begin{bmatrix} 1 \\ -3.7\ldots \end{bmatrix}$



Solutions are of the form

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$$

If $\underset{u(0)}{u_0} \neq 0$, $\underset{v(0)}{v_0} \neq 0$, then

Then we get $v = C u^k$, $k = \dfrac{\lambda_2}{\lambda_1} < 0$

since $v(t) = v_0 e^{\lambda_2 t} = v_0 e^{(\lambda_1 t)\left(\frac{\lambda_2}{\lambda_1}\right)} = v_0 \left(e^{\lambda_1 t}\right)^k$

$$= u_0^k \left(e^{\lambda_1 t}\right)^k \frac{v_0}{u_0^k} = C\, u(t)^k, \quad C = \frac{v_0}{u_0^k}$$