In order to prove Thm. 1, we shall need:

"The Fundamental Lemma" (for the Calculus of Variations)

Assume that $f : [t_0, t_1] \to \mathbb{R}$ is a continuous function and that

$$\int_{t_0}^{t_1} f(t) \cdot \mu(t) \, dt = 0$$

for all C^2-functions μ such that

$$\mu(t_0) = \mu(t_1) = 0.$$

Then $f(t) = 0$ for all $t \in [t_0, t_1]$.

Pf. (By contradiction.) Assume there exists $s \in (t_0, t_1)$ such that

$$f(s) \neq 0 \quad , \quad \text{say} \quad f(s) > 0.$$

Since f is continuous, $f(t) > 0$ for all t in some interval

$$[s-\varepsilon, s+\varepsilon]$$

We let μ be a C^2-function such that

($\mu \geq 0$, $\mu(s-\varepsilon) = \mu(s+\varepsilon) = 0$,)

$\mu > 0$ on $(s-\varepsilon, s+\varepsilon)$,

$\mu = 0$ outside $(s-\varepsilon, s+\varepsilon)$

For instance, we can take

$$\mu(t) = \begin{cases} [t-(s-\varepsilon)]^3 [(s+\varepsilon)-t]^3, & \text{if } t \in [s-\varepsilon, s+\varepsilon] \\ 0, & \text{if } t \notin [s-\varepsilon, s+\varepsilon] \end{cases}$$

Then

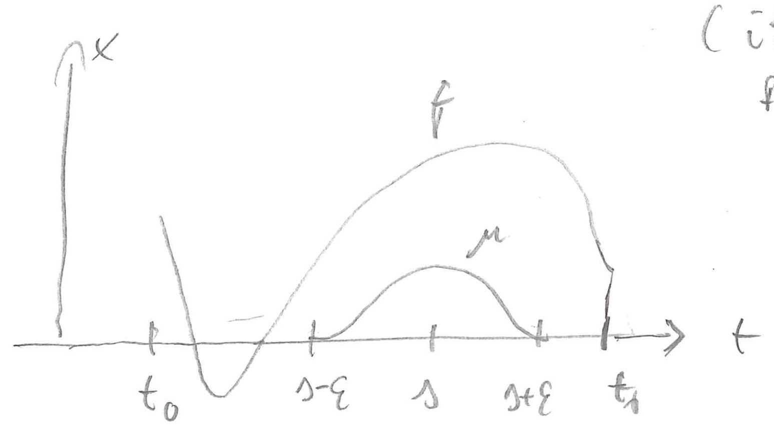
$$\int_{t_0}^{t_1} f(t)\mu(t)\,dt = \int_{s-\varepsilon}^{s+\varepsilon} f(t)\mu(t)\,dt > 0$$

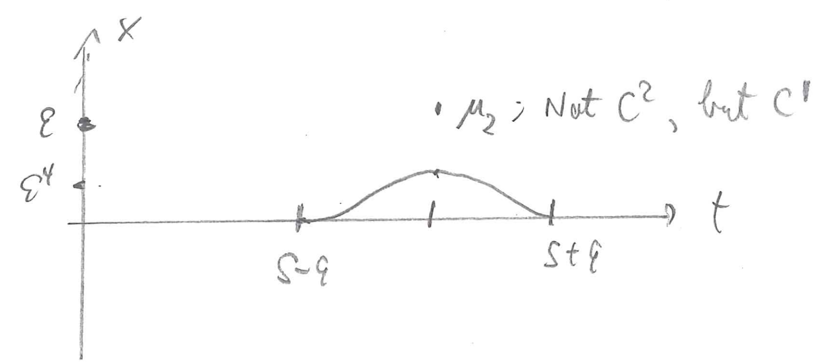
since $f(t)\mu(t) > 0$ and continuous on $(s-\varepsilon, s+\varepsilon)$.

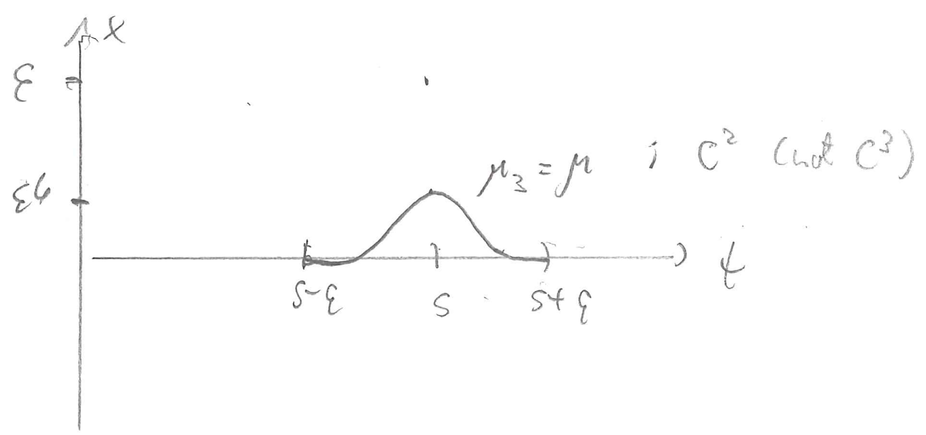
Hence we have reached a contradiction.

Thus $f(t) = 0$, for all $t \in (t_0, t_1)$, and since f is continuous, $f = 0$ on all of $[t_0, t_1]$. $\boxed{\text{}}$

(if $f \equiv 0$ on (t_0, t_1), then $f(t_0) = f(t_1) = 0$ by continuity)







We will also need to differentiate under the integral sign:

<u>Proposition.</u> Let g be a C^2-function defined on a closed rectangle $R = [a,b]\times[c,d]$ in $\mathbb{R}^2$.

Set
$$G(u) = \int_a^b g(t,u)\,dt \quad , \quad u \in [c,d].$$

Then
$$\frac{dG}{du}(u) = \int_a^b \frac{\partial g(t,u)}{\partial u}\,dt \quad , \quad \text{for all } u \in (c,d)$$

<u>Proof.</u> Let $\varepsilon > 0$. Since $\frac{\partial g}{\partial u}$ is continuous, it is also uniformly continuous on R (which is closed and bounded). Hence there is a $\delta > 0$ such that
$$\left| \frac{\partial g}{\partial u}(s,u) - \frac{\partial g}{\partial u}(t,v) \right| < \frac{\varepsilon}{b-a}$$
for all $(s,u),(t,v)$ in R with
$$\|(s,u)-(t,v)\| < \delta. \qquad (\|(a,b)\| = (a^2+b^2)^{1/2})$$

By the Mean Value Thm. there is a $\theta = \theta(t,u,v)$ between u and v, such that
$$(*) \quad g(t,v) - g(t,u) = \frac{\partial g}{\partial u}(t,\theta)\,(v-u)$$

Hence $\left| \frac{G(v)-G(u)}{v-u} - \int_a^b \frac{\partial g}{\partial u}(t,u)\,dt \right|$

$$= \left| \int_a^b \left[\frac{g(t,v)-g(t,u)}{v-u} - \frac{\partial g}{\partial u}(t,u) \right] dt \right|$$

$$\leq \int_a^b \left| \frac{\partial g}{\partial u}(t,\theta) - \frac{\partial g}{\partial u}(t,u) \right| dt \leq \int_a^b \frac{\varepsilon}{b-a}\,dt = \varepsilon$$

if $|v-u| < \delta$. Hence

$$\frac{dG(u)}{du} = \lim_{v\to u} \frac{G(v)-G(u)}{v-u} = \lim_{v\to u} \int_a^b \frac{g(t,v)-g(t,u)}{v-u}\,dt$$

$$= \int_a^b \frac{\partial g}{\partial u}(t,u)\,dt \qquad \blacksquare$$

Next, we let

$$J(x) = \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\,dt ,$$

where F and x are C^2- functions, $x(t_0)=x_0$, $x(t_1)=x_1$.
Such x are called admissible ~~(permitted)~~ functions.

<u>Proof of the "Main Theorem".</u> We consider the maximum problem:

$$(1)\begin{cases} \max\limits_{x} \int_{t_0}^{t_1} F(t,x,\dot{x})\,dt \\ x(t_0)=x_0,\ x(t_1) \end{cases}$$

Assume that x^* is a C^2-function that solves the problem (1). We wish to show that x^* satisfies the Euler equation.

Let $\mu \in C^2[t_0, t_1]$, $\mu(t_0) = \mu(t_1) = 0$.

For each real ε, the function (C^2)

$$x = x^* + \varepsilon\mu$$

is an admissible function, as $x(t_0) = x_0$, $x(t_1) = x_1$.

Since x^* maximizes J, we have

$$J(x^*) \geq J(x^* + \varepsilon\mu)$$

We let μ be fixed and study

$$I(\varepsilon) = J(x^* + \varepsilon\mu)$$

Then I has a maximum at $\varepsilon = 0$.

If I is differentiable we hence have

$$I'(0) = 0$$

Now

$$I(\varepsilon) = \int_{t_0}^{t_1} F(t, x^* + \varepsilon\mu, \dot{x}^* + \varepsilon\dot{\mu})\, dt$$

By the last proposition (with $g(t,\varepsilon) = F(t, x^* + \varepsilon\mu, \dot{x}^* + \varepsilon\dot{\mu})$)

I is differentiable and, by the Chain Rule,

$$0 = I'(0) = \int_{t_0}^{t_1} \frac{d}{d\varepsilon} F(t, x^* + \varepsilon\mu, \dot{x}^* + \varepsilon\dot{\mu}) \Big|_{\varepsilon=0} dt$$

$$= \int_{t_0}^{t_1} \left[\frac{\partial F^*}{\partial x}\mu + \frac{\partial F^*}{\partial \dot{x}}\dot{\mu} \right] dt$$

where we let

$$\frac{\partial F^*}{\partial x} = \frac{\partial F}{\partial x}(t, x^*(t), \dot{x}^*(t)),$$

and $$\frac{\partial F^*}{\partial \dot{x}} = \frac{\partial F}{\partial \dot{x}}(t, x^*(t), \dot{x}^*(t)) \;)$$

Hence

$$0 = \int_{t_0}^{t_1} \frac{\partial F^*}{\partial x}\mu \, dt + \int_{t_0}^{t_1} \overset{u}{\frac{\partial F^*}{\partial \dot{x}}} \overset{v'}{\dot{\mu}} \, dt$$

Integration by parts of the last integral gives:

$$0 = \int_{t_0}^{t_1} \frac{\partial F^*}{\partial x}\mu \, dt + \overset{=0}{\Big|_{t_0}^{t_1} \frac{\partial F^*}{\partial \dot{x}}\mu} - \int_{t_0}^{t_1} \frac{d}{dt}\left(\frac{\partial F^*}{\partial \dot{x}}\right)\mu \, dt$$

$$= \int_{t_0}^{t_1} \left[\frac{\partial F^*}{\partial x} - \frac{d}{dt}\left(\frac{\partial F^*}{\partial \dot{x}}\right)\right]\mu \, dt$$

for all such C^2-functions μ. By the Fundamental Lemma,

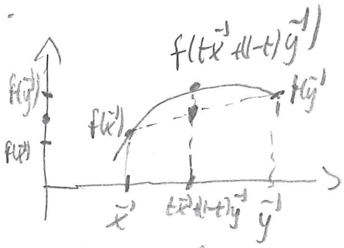
$$\frac{\partial F^*}{\partial x} - \frac{d}{dt}\left(\frac{\partial F^*}{\partial \dot{x}}\right) = 0.$$

Hence x^* satisfies the Euler-equation.

A similar argument applies to the minimum-problem.

We next prove that the Euler-eq. is sufficient for a maximum at x^* whenever F is convex in $(x, \dot{x})$, for each $t \in [t_0, t_1]$.

We shall need



__Lemma__ ("The gradient-inequality").

Let $S \subseteq \mathbb{R}^n$ be a convex (open) set, and let $f : S \to \mathbb{R}$ be C^1. If f is concave, then

$$(2) \quad f(\vec{x}) - f(\vec{y}) \le \nabla f(\vec{y})(\vec{x} - \vec{y}) = \sum_{i=1}^{n} \frac{\partial f(\vec{y})}{\partial x_i}(x_i - y_i),$$

for all $\vec{x}, \vec{y}$ in S, $\vec{x} = (x_1, \dots, x_n)$, $\vec{y} = (y_1, \dots, y_n)$.

__Remark.__ Equivalence holds above, but presently we shall only need the implication of the lemma.

__Proof.__ Assume that f is concave, and let $\vec{x}, \vec{y} \in S$.

For all $t \in (0,1)$ we have

$$t f(\vec{x}) + (1-t) f(\vec{y}) \le f(t\vec{x} + (1-t)\vec{y}); \quad t[f(\vec{x}) - f(\vec{y})] + f(\vec{y}) \le f(t\vec{x} + (1-t)\vec{y})$$

Hence $f(\vec{x}) - f(\vec{y}) \le \frac{1}{t}[f(t(\vec{x} - \vec{y}) + \vec{y}) - f(\vec{y})]$

As $t \to 0$, the right side approaches the derivative in the direction $\vec{x} - \vec{y}$. Hence

$$f(\vec{x}) - f(\vec{y}) \le f'_{\vec{x}-\vec{y}}(\vec{y}) = \nabla f(\vec{y}) \cdot (\vec{x} - \vec{y})$$

$\blacksquare$

Proof of sufficiency in the concave case:

Assume that the function

$(x,\dot{x}) \longmapsto F(t,x,\dot{x})$

is concave for each fixed t in $[t_0, t_1]$.

Suppose further that x^* is admissible and satisfies the Euler-eq.

We shall prove that x^* maximizes the integral $J(x)$. If x is any admissible (permitted) C^2-function with $x(t_0)=x_0$, $x(t_1)=x_1$, then the above Gradient-ineq. yields

$$F(t,x,\dot{x}) - F(t,x^*,\dot{x}^*)$$

$$\leq \frac{\partial F^*}{\partial x}(x-x^*) + \frac{\partial F^*}{\partial \dot{x}}(\dot{x}-\dot{x}^*)$$

$$= \frac{d}{dt}\left(\frac{\partial F^*}{\partial \dot{x}}\right)(x-x^*) + \frac{\partial F^*}{\partial \dot{x}}(\dot{x}-\dot{x}^*) \quad \text{(by the Euler-eq.)}$$

$$= \frac{d}{dt}\left[\frac{\partial F^*}{\partial \dot{x}}(x-x^*)\right] \quad \text{(the product rule)}$$

Integration of this ineq. gives

$$J(x) - J(x^*) \leq \int_{t_0}^{t_1} \frac{d}{dt}\left[\frac{\partial F^*}{\partial \dot{x}}(x-x^*)\right] dt$$

$$= \left[\frac{\partial F^*}{\partial \dot{x}}(x-x^*)\right]_{t_0}^{t_1} = 0$$

since $x(t_1) - x^*(t_1) = x_1 - x_1 = 0$, $x(t_0) - x^*(t_0) = 0$. Hence $J(x) \leq J(x^*)$ for all admissible x, and $J(x^*)$ is the maximum. $\boxed{\Xi}$ 1 20.02.12 , 23.02.16