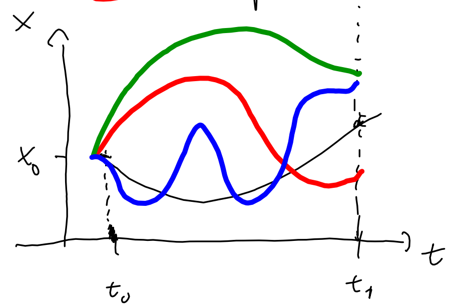
Other endpoint conditions.

x_0 is given, fixed real number.



We will consider two cases

(a) $x(t_0)=x_0$ fixed, $x(t_1)$ free

and

(b) $x(t_0)=x_0$ fixed, $x(t_1) \geq x_1$, x_1 given and fixed

Theorem 2. Let t_0, t_1, x_0 be given, fixed real numbers, and F is a given C^2-function of 3 variables, defined over some region in $\mathbb{R}^3$. Consider

$$(*) \qquad J(x) = \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\,dt, \quad x \in C^2[t_0, t_1]$$

(a) A necessary condition for x^* to solve the max (or min.) problem

$$\max_x J(x), \ x(t_0)=x_0, \ x(t_1) \text{ is free},$$

is that x^* satisfies the Euler equation

$$(E) \qquad \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \qquad \left(\text{i.e. } \frac{\partial F}{\partial x} = \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right)\right),$$

and the "transversality" condition

$$(T) \qquad \left(\frac{\partial F}{\partial \dot{x}}\right)_{t=t_1} = 0 \qquad \left(\frac{\partial F}{\partial \dot{x}}(t_1, x^*(t_1), \dot{x}^*(t_1)) = 0\right)$$

(b) If x_1 is given, then x^* solves the problem

$$\max_x J(x), \quad x(t_0) = x_0, \quad x(t_1) \geq x_1$$

only if (E) is satisfied by x^* and also the condition

$$(T') \qquad \left(\frac{\partial F}{\partial \dot{x}}\right)_{t=t_1} \leq 0 \qquad (= 0 \text{ if } x^*(t_1) > x_1)$$

hold true.

If $(x, \dot{x}) \mapsto F(t, x, \dot{x})$ is concave (resp. convex) for each fixed t in $[t_0, t_1]$, then every x^* that satisfies (E) and (T) ((T') for (b)) will solve the problem.

Proof. (For $x(t_1)$ free, and the Max. problem)
All the admissible x that have the same value $x^*(t_1)$ at t_1 as x^* does, must satisfy

$$J(x) \le J(x^*)$$

so x^* is optimal among all those functions x. But now we are reduced to the case of fixed endpoints

$$x(t_1) = x^*(t_1).$$

Hence Theorem 1 applies and yields that x^* must satisfy (E).
In the proof of Thm. 1 (when we derived (E)) we considered

$$I(\varepsilon) = \int_{t_0}^{t_1} F(t, x^* + \varepsilon\mu, \dot{x}^* + \varepsilon\dot{\mu})\,dt$$

After integration by parts and differentiation under the integral sign, we found that

$$0 = I'(0) = \int_{t_0}^{t_1} \underbrace{\left[\frac{\partial F^*}{\partial x} - \frac{d}{dt}\left(\frac{\partial F^*}{\partial \dot{x}}\right)\right]}_{0}\mu(t)\,dt + \left[\left(\frac{\partial F^*}{\partial \dot{x}}\right)\mu(t)\right]_{t_0}^{t_1}$$

where (E) for x^* holds; so that, when choosing μ so that $\mu(t_1) \ne 0$, $\mu(t_0) = 0$

$$0 = \left(\frac{\partial F^*}{\partial \dot{x}}\right)_{t = t_1} \qquad (T)$$

Sufficiency: Suppose that $F(t, x, \dot{x})$ concave in $(x, \dot{x})$ for each $t \in [t_0, t_1]$ and further, that x^* is an admissible function which satisfies (E) and (T).
First, we can argue as in the proof of sufficiency in Theorem 1:
By the Gradient ineq. (G) and (E), we have

$$J(x) - J(x^*) = \int_{t_0}^{t_1} [F(t, x, \dot{x}) - F(t, x^*, \dot{x}^*)]\,dt$$

$$\overset{(G)}{\le} \int_{t_0}^{t_1} \Big[\underbrace{\frac{\partial F^*}{\partial x}}_{(E)}\cdot(x - x^*) + \frac{\partial F^*}{\partial \dot{x}}(\dot{x} - \dot{x}^*)\Big]\,dt$$

$$\overset{(E)}{=} \int_{t_0}^{t_1} \left[\frac{d}{dt}\left(\frac{\partial F^*}{\partial \dot{x}}\right)\cdot(x - x^*) + \frac{\partial F^*}{\partial \dot{x}}(\dot{x} - \dot{x}^*)\right]dt$$

$$= \int_{t_0}^{t_1} \frac{d}{dt}\left[\frac{\partial F^*}{\partial \dot{x}}(x - x^*)\right]dt = \left[\frac{\partial F^*}{\partial \dot{x}}(x - x^*)\right]_{t_0}^{t_1}$$

$$= \left[\frac{\partial F^*}{\partial \dot{x}}(x - x^*)\right]\Big|_{t = t_1} \overset{(T)}{=} \underline{0}$$

So $J(x) - J(x^*) \le 0$ and <u>$J(x^*)$ must be maximal.</u> ∎

Example. Solve

$$\min \int_0^1 \underbrace{(x^2+\dot{x}^2)}_{F(t,x,\dot{x})}\,dt \quad , \quad x(0)=1 \;,\; x(1) \text{ free.}$$

The Euler eq. becomes

$$(r^2-1=0,\ r=\pm 1)$$

(E) $\ddot{x} - x = 0$ $\quad (\cancel{xyz})$

$x(t) = Ae^t + Be^{-t}$

$x(0) = A + B = 1, \quad B = 1 - A,$

$\boxed{x(t) = Ae^t + (1-A)e^{-t}}$

Must have

(T) $0 = \left(\frac{\partial F}{\partial \dot{x}}\right)_{t=1} = (2\dot{x})_{t=1} = 2\dot{x}(1) = 0$

$= 2\left[Ae^t + (1-A)(-1)e^{-t}\right]_{t=1} = 2\left[Ae + (1-A)(-1)\cdot e^{-1}\right]$

So $A = \dots = \frac{1}{e^2+1} = \frac{e^{-1}}{e+e^{-1}}$

$x^*(t) = \frac{1}{e+e^{-1}}\left[e^{t-1} + e^{1-t}\right]$

Here $(\dot{x}, x) \mapsto x^2 + \dot{x}^2$ is convex, x^* minimizes $J(x)$.

Systems of differential equations

Ex. t: independent variable; x, y dependent variables.

$$\begin{cases} f(t, x, y, x', y') = 0 \\ g(t, x, y, x', y') = 0 \end{cases}$$

← 1st order system

f, g given functions (of 5 variables)

Typically, the number no. of equations will be the same as the number of dependent variables.

Ex. Consider a 2nd-order system of the form:

$$(1) \begin{cases} x_1'' = f_1(t, x_1, x_2, x_1', x_2') \\ x_2'' = f_2(t, x_1, x_2, x_1', x_2') \end{cases}$$

Such systems can be transformed into an equivalent system of 1st order equations the following way:

$$y_1 = x_1, \quad y_2 = y_1' = x_1'$$

$$z_1 = x_2, \quad z_2 = z_1' = x_2'$$

Hence we get the first order system:

$$(2) \begin{cases} y_1' = y_2 \\ y_2' = f_1(t, y_1, z_1, y_2, z_2) \\ z_1' = z_2 \\ z_2' = f_2(t, y_1, z_1, y_2, z_2) \end{cases}$$

4 dependent variables (y_1, y_2, z_1, z_2)
1 indep. var. (t)

<u>Ex.</u> The n-th order eq.

(*) $x^{(n)} = f(t, x, x', x'', \dots, x^{(n-1)})$

can be reduced to a system of 1st order eq.:

$$(**)\begin{cases} x_1 = x \\ x_1' = x_2 \\ x_2' = x_3 \\ \vdots \\ x_{n-1}' = x_n \\ x_n' = f(t, x_1, x_2, \dots, x_n) \end{cases}$$

Here (*) and (**) are <u>equivalent</u> systems
(they have exactly the same solutions)

<u>Ex.</u> The 3rd order eq.

$x^{(3)} + 3x'' + 2x' - 5x = \sin(2t)$

can be written as in (*):

$x^{(3)} = 5x - 2x' - 3x'' + \sin(2t)$ $\quad (= f(t, x, x', x''))$

Hence we let

$$\begin{cases} x_1 = x, \\ x_2 = x_1' \\ x_3 = x_2' \\ x_3' = 5x_1 - 2x_2 - 3x_3 + \sin(2t) \end{cases}$$

Ex. (Nonlinear)

$$x'' = x^3 + (x')^3$$

yields the 1st order system ($x_1 = x$, $x_2 = x_1' = x'$)

$$\begin{cases} x_1' = x_2 \\ x_2' = x_1^3 + x_2^3 \end{cases}$$

for which effective numerical methods apply

(see EP: 6.3 p. 453 ; 6.4 p. 464–479)

(Based on Simpson's formula)

Ex. Shall transform 2nd-order system

$$2x'' = -6x + 2y$$

$$y'' = 2x - 2y + 4\sin(3t)$$

into an equivalent 1st order system: Define

$x_1 = x$, $x_2 = x' = x_1'$

$y_1 = y$, $y_2 = y' = y_1'$

This yields

$$\begin{cases} x_1' = x_2 \\ 2x_2' = -6x_1 + 2y_1 \quad \text{or)} \quad x_2' = -3x_1 + y_1 \\ y_1' = y_2 \\ y_2' = 2x_1 - 2y_1 + 4\sin(3t) \end{cases}$$

$$\begin{bmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4\sin 3t \end{bmatrix}$$

4 indep. variables x_1, x_2, y_1, y_2, 4 equations (linear)