March 29.16

Independence of General Solutions.

Def. The vector valued functions $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n$ are <u>linearly dependent</u> on an interval I if there exists constants

$$c_1, c_2, \dots, c_n,$$

not all 0, such that

$$(*)\quad c_1\vec{X}_1 + c_2\vec{X}_2 + \dots + c_n\vec{X}_n = \vec{0}$$

Otherwise they are called <u>lin. independent</u>, that is, if $(*)$ holds for constants $c_1, \dots, c_n$, then $c_1 = c_2 = \dots = c_n = 0$

If $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n$ are solutions of a linear system

$$\vec{X}' = P(t)\vec{X},$$

write $\vec{X}_k(t) = \begin{bmatrix} X_{1k}(t) \\ \vdots \\ X_{nk}(t) \end{bmatrix}$, $t \in I$, $1 \le k \le n$

Set $P(t) = [p_{ij}(t)]$ $\quad 1 \le i, j \le n$, $t \in I$,

Then define the Wronsky determinant by

$$W(t) = \begin{vmatrix} X_{11}(t) & \cdots & X_{1n}(t) \\ X_{21}(t) & \cdots & X_{2n}(t) \\ \vdots & & \vdots \\ X_{n1}(t) & \cdots & X_{nn}(t) \end{vmatrix}, \quad t \in I$$

$$= |\vec{X}_1\ \vec{X}_2 \dots \vec{X}_n|_{(t)} = W(\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n)(t)$$

The following theorem is similar to the one for a single n-th order equation (See 2.2, Thm 3) p. 119.

The proofs are essentially the same, using Abel's formula for the Wronskian.

<u>Theorem 2</u> (Wronskian of Solutions).

Assume that $\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n$ are solutions of the linear, homogeneous system $\vec{X}' = P(t)\vec{X}$, $t \in I$,

and the matrix $P(t)$ is continuous on I, then:

(1) $\vec{X}_1, \dots, \vec{X}_n$ are linearly indep. on I, implies that $W(\vec{X}_1, \vec{X}_2, \dots, \vec{X}_n) \neq 0$ for all t in I

(2) If $\vec{X}_1, \dots, \vec{X}_n$ are linearly dependent on I, then $W(\vec{X}_1, \dots, \vec{X}_n) = 0$ for all t in I.

Example. Let

$$\vec{X}_1(t) = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \quad \vec{X}_2(t) = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}, \quad \vec{X}_3(t) = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$$

Then $\vec{X}_1, \vec{X}_2, \vec{X}_3$ are solutions of

$$\frac{d\vec{X}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \vec{X} = P\vec{X}.$$

For example

$$\begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} = e^t \begin{bmatrix} 6-4 \\ -2+6-2 \\ 0-2+3 \end{bmatrix} = e^t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}$$

Hence $\vec{X}_1'(t) = e^t \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = P\vec{X}_1(t)$

$(t \in \mathbb{R})$

The Wronskian of these solutions:

$$W(t) = \begin{vmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{vmatrix} = e^t \cdot e^{3t} \cdot e^{5t} \begin{vmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{vmatrix}$$

$$= e^{9t} \cdot 2 \cdot 2 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix} = 4e^{9t}[-1-1\cdot 2+1\cdot(-1)]$$

$$= \underline{-16e^{9t}} \neq 0$$

for all t. Hence $\vec{X}_1, \vec{X}_2, \vec{X}_3$ are linearly independent solutions.

Theorem 3 (General solutions, homogeneous systems)

I : open interval

$P(t)$: $n\times n$ matrix for all $t\in I$, continuous on I.

Let $\vec{x}_1, \vec{x}_2, \ldots \vec{x}_n$ be n linearly independent solutions of

$$(*) \quad \vec{x}\,' = P(t)\,\vec{x} \quad \text{on } I.$$

Then any solution $\vec{x}$ of $(*)$ can be expressed as

$$\vec{x} = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n \quad (\text{on } I)$$

for some numbers $c_1, c_2, \ldots, c_n$.

Proof. Let $a\in I$ be fixed.
Must show:

CLAIM: There are $c_1, c_2, \ldots, c_n$ such that the solution

$$\vec{y}(t) = c_1\vec{x}_1(t) + \cdots + c_n\vec{x}_n(t)$$

has the same initial value at a as does the given solution $\vec{x}(t)$:

$$(**) \quad \vec{y}(a) = \vec{x}(a).$$

Let $Q(t) = [\vec{x}_1(t) \ldots \vec{x}_n(t)]$ ($n\times n$ matrix)

Let $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. Then $(**)$ can be written as

$$Q(a)\,\vec{c} = \vec{x}(a)$$

Since $\vec{X}_1, \dots, \vec{X}_n$ are linearly independent, the Wronskian

$$W(a) = |Q(a)| \neq 0$$

Hence $Q(a)$ has an inverse matrix $Q(a)^{-1}$. Thus, since

$$Q(a)\vec{c} = \vec{X}(a) \quad ,$$

we have that

$$\vec{c} = Q(a)^{-1} Q(a) \vec{c} = \underbrace{Q(a)^{-1} \vec{X}(a)}$$

satisfies $(**)$. Hence such $\vec{c}$ exists, as claimed.

Finally, the solutions $\vec{Y}(t)$ and $\vec{X}(t)$ have the same initial value at $t = a$.

From the (existence and) uniqueness theorem (Section 5.1),

$$\vec{X} = \vec{Y} \quad \text{on } I. \qquad \boxtimes$$

We may try to use the above theorem to find a general solution of (*) $\boxed{\vec{x}' = P\vec{x}.}$

First, we prove:

Theorem (Eigenvalue Solutions)

Let λ be an eigenvalue with a corresponding eigenvector $\vec{v}$ of P in (*). Then

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

is a (nontrivial) solution of (*).

Proof. Suppose $P\vec{v} = \lambda\vec{v}$.

Then $(\vec{x}(t) = e^{\lambda t}\vec{v})$

$$\vec{x}'(t) = \frac{d}{dt}(\vec{x}(t)) = \lambda e^{\lambda t}\vec{v}$$
$$= \lambda\vec{x}(t) = e^{\lambda t}P\vec{v}$$
$$= P(e^{\lambda t}\vec{v}) = P(\vec{x}(t))$$

Thus $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a solution of $\vec{x}' = P\vec{x}$. ∎

If we can find n linearly independent eigenvectors of P, $\vec{v}_1, \dots, \vec{v}_n$, with eigenvalues $\lambda_1, \dots, \lambda_n$ (respectively), then we obtain a general solution by

$$\vec{x}(t) = c_1\vec{x}_1(t) + \dots + c_n\vec{x}_n(t),\ t \in I,$$
$$\vec{x}_k(t) = e^{\lambda_k t}\cdot\vec{v}_k \quad (1 \le k \le n)$$

However, this does not always work:

Example Let $P = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

The only eigenvalue is $\lambda = 1$.

Now $P\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \Leftrightarrow \begin{bmatrix} x+y \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$\Leftrightarrow y = 0$. Hence $\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ generates the one-dimensional eigenspace.

Complex Eigenvalues.

The above eigenvalue method also applies to complex eigenvalues

$\lambda = p + iq \quad , \quad p, q \in \mathbb{R}$

However, the eigenvectors will then usually be complex valued.

We will always assume that the matrix P of the system is real (has only real entries, $P = [p_{ij}]_{1 \le i,j \le n} \quad p_{ij} \in \mathbb{R}$)

Thus the characteristic equation

$$p(\lambda) = |P - \lambda I| = 0$$

will have only real coefficients: They are polynomials in the p_{ij}-s

Example $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$|P - \lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc)$$

$$= \lambda^2 - (\mathrm{Tr}\, P)\lambda + \det P$$

Hence

$$p(\lambda) = 0 \iff p(\bar{\lambda}) = \overline{p(\lambda)} = 0 ,$$

So λ is an eigenvalue $\iff \bar{\lambda}$ is an eigenvalue

$\vec{v}$: eigenvector of λ

Then , $P\vec{v} = \lambda\vec{v}$, so that

$$\vec{0} = (P - \lambda I)\vec{v} = \overline{(P - \lambda I)\vec{v}}$$

$$= (\overline{P} - \overline{\lambda I})\overline{\vec{v}}$$

$$= (P - \overline{\lambda} I)\overline{\vec{v}}$$

Hence $\overline{\vec{v}}$ is an eigenvector with eigenvalue $\overline{\lambda}$ (and conversely)

Therefore,

if $\vec{X}(t) = e^{\lambda t}\vec{v}$ is a solution of $\vec{X}' = P\vec{X}$,

corresp. to the eigenvector/eigenvalue $\vec{v}/\lambda$, then

$$\overline{\vec{X}(t)} = e^{\overline{\lambda} t}\,\overline{\vec{v}}$$

CLAIM: If $\vec{X}(t)$ is a solution of (*) $\vec{X}'(t) = P\vec{X}(t)$,

then $\operatorname{Re}\vec{X}(t)$ and $\operatorname{Im}\vec{X}(t)$ are also solutions of (*).

Proof of CLAIM:

Write $\vec{x}_1(t) = \mathrm{Re}\,\vec{x}(t)$, $\vec{x}_2(t) = \mathrm{Im}\,\vec{x}(t)$.

Hence $\vec{x}(t) = \vec{x}_1(t) + i\,\vec{x}_2(t)$

Then $\mathrm{Re}\,(P\vec{x}(t)) = \mathrm{Re}\,(P\,(\vec{x}_1(t) + i\,\vec{x}_2(t)))$

$= \mathrm{Re}\,(P\vec{x}_1(t) + i\,P\vec{x}_2(t))$

$= P\vec{x}_1(t)$

Thus:

$\mathrm{Re}\,P\vec{x}(t) = P\vec{x}_1(t) = P(\mathrm{Re}\,\vec{x}(t))$

Hence

$\vec{0} = \mathrm{Re}\,(\vec{x}\,'(t) - P\vec{x}(t))$

$= \mathrm{Re}\,\vec{x}\,'(t) - \mathrm{Re}\,P\vec{x}(t)$

$= \mathrm{Re}\,[\vec{x}_1\,'(t) + i\,\vec{x}_2\,'(t)] - P\vec{x}_1(t)$

$= \vec{x}_1\,'(t) - P\vec{x}_1(t)$

Hence $\mathrm{Re}\,\vec{x}(t) = \vec{x}_1(t)$ is a solution of $(*)$.

Similarly, $\mathrm{Im}\,\vec{x}(t) = \vec{x}_2(t)$ is also a solution of $(*)$.

Next, observe that

$$\mathrm{Im}(\overline{\vec{x}}) = \mathrm{Im}\,(\vec{x}_1 - i\vec{x}_2) = -\vec{x}_2$$
$$= -\,\mathrm{Im}\,\vec{x}$$

and $\mathrm{Re}(\overline{\vec{x}}) = \vec{x}_1 = \mathrm{Re}\,\vec{x}$.

Hence it suffices to find and use the two solutions

$$\vec{x}_1 = \mathrm{Re}\,\vec{x}\ , \quad \vec{x}_2 = \mathrm{Im}\,\vec{x},$$

associated to a complex eigenvalue $\lambda = p + iq$: <u>We do not need to consider $\overline{\lambda}$ and $\overline{\vec{x}}$!</u>