We had

$$(E) \quad e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}, \quad \text{for any } n\times n \text{ matrix } A$$

As for real valued power series, the series (E) can be differentiated termwise with respect to (w.r.t.) t:

$$\frac{d}{dt} e^{tA} = \sum_{n=1}^{\infty} \frac{t^{n-1}A^n}{(n-1)!} = A \sum_{n=1}^{\infty} \frac{t^{n-1}A^{n-1}}{(n-1)!}$$

$$= A \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = A e^{tA}$$

Hence e^{tA} satisfies the matrix diff. equation

$$X' = AX$$

In addition, we have $e^{tA}\big|_{t=0} = e^{0\cdot A} = I$

Since e^{tA} is invertible with inverse

$$e^{-tA} = (e^{tA})^{-1},$$

$\Phi(t) = e^{tA}$ is a fundamental matrix for the linear homogeneous system

$$(*) \quad \vec{x}' = A\vec{x}$$

Moreover, the columns of e^{tA} form n linearly independent solutions $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ of $(*)$

For any initial value condition

$$(**) \quad \vec{x}'(0) = \vec{x}_0$$

of $(*)$ we have the unique solution

$$\vec{x}(t) = \Phi(t)\,\Phi(0)^{-1}\vec{x}_0 \qquad (\Phi \text{ any fundamental matrix of } (*))$$

If $\Phi(t) = e^{tA}$, then $\Phi(0) = I$, so $\Phi(0)^{-1} = I$

and

$$\vec{x}(t) = e^{tA}\vec{x}_0$$

is the unique solution of $(*)$ and $(**)$.

Suppose $\Psi(t)$ is any fundamental matrix for $(*)$, we also have $\vec{x}(t) = \underbrace{\Psi(t)\,\Psi(0)^{-1}}\vec{x}_0$

is the unique solution of $(*)$ and $(**)$, for any $\vec{x}_0 \in \mathbb{R}^n$

This means
$$e^{tA}\vec{x}_0 = \Psi(t)\Psi(0)^{-1}x_0, \quad \text{for all } \vec{x}_0 \text{ in } \mathbb{R}^n.$$
Hence the linear operators are equal:
$$(***) \quad \boxed{e^{tA} = \Psi(t)\Psi(0)^{-1}} \qquad (\text{for all } t \in I)$$
This yields a method of finding e^{tA}, given any fundamental matrix $\Psi(t)$ of $(*)$

<u>5.7</u> (3)(a) $\vec{x}\,' = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}\vec{x}$, $\vec{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

We find a fundamental matrix $\Phi(t)$ of (a) and then use $(***)$ above:

<u>Eigenvalues</u>:
$$0 = |\lambda I - A| = (-1)^2|A - \lambda I| = \begin{vmatrix} \lambda-2 & 5 \\ -4 & \lambda+2 \end{vmatrix} = \lambda^2 - 4 + 20 = \lambda^2 + 16$$
$$\underline{\lambda = \pm 4i} \qquad (\text{imaginary eigenvalues})$$

$(2-4i)(2+4i)(-1) = -(4+16) = -20$

<u>Eigenvectors for $\lambda = 4i$</u>: $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$
$$(A - 4iI) = \begin{bmatrix} 2-4i & -5 \\ 4 & -2-4i \end{bmatrix} \sim \begin{bmatrix} 2-4i & -5 \\ 4(2-4i) & -20 \end{bmatrix} \sim \begin{bmatrix} 2-4i & -5 \\ 0 & 0 \end{bmatrix}$$
Hence
$$(2-4i)a = 5b.$$
Let $b = 2-4i$. Then $a = 5$; $\vec{v} = \begin{bmatrix} 5 \\ -2-4i \end{bmatrix}$,
$$\vec{v} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} + i\begin{bmatrix} 0 \\ -4 \end{bmatrix}$$
Associated complex solution:
$$\vec{x}(t) = e^{4ti}\left(\begin{bmatrix} 5 \\ +2 \end{bmatrix} + i\begin{bmatrix} 0 \\ -4 \end{bmatrix}\right) = (\cos 4t + i\sin 4t)\left(\begin{bmatrix} 5 \\ +2 \end{bmatrix} + i\begin{bmatrix} 0 \\ -4 \end{bmatrix}\right)$$
$$= \begin{bmatrix} 5\cos 4t \\ +2\cos 4t \end{bmatrix} - \begin{bmatrix} 0 \\ -4\sin 4t \end{bmatrix} + i\left(\begin{bmatrix} 0 \\ -4\cos 4t \end{bmatrix} + \begin{bmatrix} 5\sin 4t \\ +2\sin 4t \end{bmatrix}\right)$$
$$= \begin{bmatrix} 5\cos 4t \\ +2\cos 4t + 4\sin 4t \end{bmatrix} + i\begin{bmatrix} 5\sin 4t \\ -4\cos 4t + 2\sin 4t \end{bmatrix}$$

Linearly independ. real solutions:

$$\vec{x}_1(t) = \begin{bmatrix} 5\cos 4t \\ 2\cos 4t + 4\sin 4t \end{bmatrix}, \quad \vec{x}_2(t) = \begin{bmatrix} 5\sin 4t \\ 2\sin 4t - 4\cos 4t \end{bmatrix}$$

A fundamental matrix

$$\Phi(t) = \left[\vec{x}_1(t) \ \ \vec{x}_2(t)\right] = \begin{bmatrix} 5\cos 4t & 5\sin 4t \\ 2\cos 4t + 4\sin 4t & 2\sin 4t - 4\cos 4t \end{bmatrix}$$

$$\Phi(0) = \begin{bmatrix} 5 & 0 \\ 2 & -4 \end{bmatrix}, \quad \Delta = |\Phi(0)| = -20$$

$$\Phi(0)^{-1} = -\frac{1}{20}\begin{bmatrix} -4 & 0 \\ -2 & 5 \end{bmatrix}$$

$$\Phi(0)^{-1}\vec{x}(0) = -\frac{1}{20}\begin{bmatrix} -4 & 0 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{20}\begin{bmatrix} 0 \\ 5 \end{bmatrix} = \underline{\frac{1}{4}\begin{bmatrix} 0 \\ -1 \end{bmatrix}}$$

The unique solution is

$$\vec{x}(t) = \Phi(t)\Phi(0)^{-1}\vec{x}_0 = \frac{1}{4}\begin{bmatrix} 5\cos 4t & 5\sin 4t \\ 2\cos 4t + 4\sin 4t & 2\sin 4t - 4\cos 4t \end{bmatrix}\begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$= \underline{\frac{1}{4}\begin{bmatrix} -5\sin 4t \\ 4\cos 4t - 2\sin 4t \end{bmatrix}}$$

<u>5.7</u> (21) $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. Here

$$A^2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$ so A is (2-step)

nilpotent. Hence

$$e^{tA} = I + tA + 0 + \dots = \underline{\begin{bmatrix} 1+t & -t \\ t & 1-t \end{bmatrix}}$$

<u>Obligatory / compulsory problems:</u>

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5.7 (25)

$$A = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix} = 2I + N \quad , \quad N = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} , \quad N^2 = 0$$

$$e^{tA} = e^{t(2I+N)} = e^{2tI + tN} = e^{2tI} e^{tN}$$

(since $(2I)N = N(2I)$) $= e^{2t} I \cdot e^{tN}$

$$= e^{2t}(I + tN) = e^{2t} \cdot \begin{bmatrix} 1 & t5 \\ 0 & 1 \end{bmatrix} = \Phi(t)$$

Since $e^{0 \cdot A} = I = \Phi(0)$, the unique solution of

$$\vec{x}' = A\vec{x} \quad , \quad \vec{x}(0) = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

is $\vec{x}(t) = e^{tA} \cdot \vec{x}(0) = e^{2t}\begin{bmatrix} 1 & t5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = e^{2t}\begin{bmatrix} 4+35t \\ 7 \end{bmatrix}$

Example $A = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix}$ $\quad b \neq 0, b \neq 1$.

$$A = \begin{bmatrix} 1 & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & a \\ 0 & 0 \end{bmatrix} = D+N \quad \text{but} \quad DN \neq ND$$

Hence we have $e^{t(D+N)} \neq e^{tD} e^{tN}$

Eigenvalues : $\lambda = 1, b$

Eigenvectors $\begin{bmatrix} x \\ y \end{bmatrix}$,

$\lambda = 1$ $\quad \lambda I - A = \begin{bmatrix} 0 & -a \\ 0 & 1-b \end{bmatrix}$, $y = 0$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ spans eigenspace V_1.

$\lambda = b$ $\quad \lambda I - A = \begin{bmatrix} b-1 & -a \\ 0 & 0 \end{bmatrix}$, $(b-1)x = ay$, Let $x = a, y = b-1$

$\vec{v}_2 = \begin{bmatrix} a \\ b-1 \end{bmatrix}$ spans eigenspace V_b

Fundamental matrix

$$\Phi(t) = \begin{bmatrix} e^t & a e^{bt} \\ 0 & (b-1)e^{bt} \end{bmatrix} , \quad \Phi(0) = \begin{bmatrix} 1 & a \\ 0 & b-1 \end{bmatrix}$$

$$|\Phi(0)| = b-1 , \quad \Phi(0)^{-1} = \frac{1}{b-1}\begin{bmatrix} b-1 & -a \\ 0 & 1 \end{bmatrix}$$

So $e^{tA} = \Phi(t)\Phi(0)^{-1} = \frac{1}{b-1}\begin{bmatrix} e^t & a(e^{bt} - e^t) \\ 0 & (b-1)e^{bt} \end{bmatrix}$

[The matrices

$A = \begin{bmatrix} a & b \\ 0 & 1 \end{bmatrix}$ $(a \neq 0)$ form the group of affine motions of $\mathbb{R}$, also called the "$ax+b$ group"

$$A\begin{bmatrix} x \\ 1 \end{bmatrix} = ax + b \quad]$$

5.7 (29)

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I + N, \quad N^4 = 0$$

N is 4-step nilpotent

$$N^3 = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} N = \begin{bmatrix} 0 & 0 & 12 & 12 \\ 0 & 0 & 0 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 24 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$$e^{tA} = e^t I \left(I + tN + \tfrac{1}{2} t^2 N^2 + \tfrac{1}{6} t^3 N^3 \right)$$

$$= e^t \begin{bmatrix} 1 & 2t & 3t + 6t^2 & 4t + 6t^2 + 4t^3 \\ 0 & 1 & 6t & 3t + 6t^2 \\ 0 & 0 & 1 & 2t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ has the unique solution

$$\vec{x}(t) = e^{tA} \vec{x}(0) = e^t \begin{bmatrix} 1 + 9t + 12t^2 + 4t^3 \\ 1 + 9t + 6t^2 \\ 1 + 2t \\ 1 \end{bmatrix}$$

(34) $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & 0 \\ 0 & -4 \end{bmatrix} = -2^2 \cdot I$$

$A^3 = -2^2 A$, $A^4 = -2^2 A^2 = 2^4 \cdot I$, $A^5 = 2^4 A$

$A^6 = 2^4 \cdot A^2 = -2^6 I$

. . . . . .

$A^{2n} = (-1)^n 2^{2n} \cdot I$, $A^{2n+1} = (-1)^n 2^{2n} A$

(can be proved by induction)

Hence

$$e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} = \left[\sum_{n=0}^{\infty} \frac{(2t)^{2n}}{(2n)!}(-1)^n\right] \cdot I + \left[\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1} 2^{2n}}{(2n+1)!}\right] A$$

$$= (\cos 2t) I + \left(\frac{1}{2} \sin 2t\right) A$$

Solving $^{(*)}$ $\vec{x}' = A\vec{x}$:

$$e^{tA} = \begin{bmatrix} \cos 2t & 0 \\ 0 & \cos 2t \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & 2\sin 2t \\ -2\sin 2t & 0 \end{bmatrix} = \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$$

Hence a general solution of $(*)$ is

$$\vec{x}(t) = c_1 \begin{bmatrix} \cos 2t \\ -\sin 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t \\ \cos 2t \end{bmatrix}$$

(c_1, c_2 arbitrary real numbers)