Mat 2440 May 4, 2016 Problem Set 11

SSS 12.2.1/SHSS 9.2.1

$\max_u \int_0^2 [e^t x(t) - u(t)^2]\,dt$, $\dot{x}(t) = -u(t)$, $x(0) = 0$, $x(2)$ is free, $u(t) \in \mathbb{R}$, $t \in [0,2]$.

Since $x(2)$ is free, $p(2) = 0$, so this is a Normal Problem. Hence

Hence the Hamiltonian is

$$H(t, x, u, p) = e^t x - u^2 - pu$$

which is concave of (x, u) for each t, by the 2nd derivative test:

$$\frac{\partial^2 H}{\partial u^2} = -2 < \underline{0}, \quad \frac{\partial^2 H}{\partial x^2} = 0 \leq \underline{0}, \quad \frac{\partial^2 H}{\partial x \partial u} = \frac{\partial^2 H}{\partial u \partial x} = 0$$

Hence $\Delta_H = \begin{vmatrix} \frac{\partial^2 H}{\partial u^2} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial x \partial u} & \frac{\partial^2 H}{\partial x^2} \end{vmatrix} = (-2)\cdot 0 - 0^2 = 0 \geq \underline{0}$

Hence we have concavity. (Alternative: $\varphi_t(x) = e^t x$ and $\psi_t(u) = -u^2 - pu$ are concave of x, resp. of u, hence their sum is concave of (x, u))

Therefore, Mangasarian's Thm. applies, so any admissible pair (x^*, u^*) that satisfies the conditions of the Maximum Principle is optimal.

Suppose (x^*, u^*) is an optimal pair. (a) Then $u = u^*(t)$ must maximize $H(t, x^*(t), u, p(t))$ as a function of u, for each $t \in [0,2]$. Here

$$\frac{\partial H^*}{\partial u} = -2u^*(t) - p(t) = 0 \iff u^*(t) = -\tfrac{1}{2} p(t), \quad t \in [0,2].$$

This is a maximum since H was concave of (x, u), hence also of u $\left(\frac{\partial^2 H^*}{\partial u^2} = -2 < 0\right)$. Hence $u^*(t) = -\frac{1}{2} p(t)$, $t \in [0,2]$.

(b) Furthermore, $\frac{\partial H^*}{\partial x} = -\dot{p}(t)$, $\dot{p}(t) = -e^t$, $p(t) = -e^t + A$ (iii')

As $x(2)$ is free, $p(2) = 0$, $\underline{A = e^2}$

So $\underline{p(t) = e^2 - e^t}$

$\underline{u^*(t) = -\tfrac{1}{2} p(t) = \tfrac{1}{2}(e^t - e^2)}$

By the "equation of state":

$$\dot{x}^*(t) = -u^*(t) = \tfrac{1}{2}(e^2 - e^t)$$

$$x^*(t) = \tfrac{1}{2}(e^2 t - e^t) + B,$$

where $x^*(0) = 0 \iff \tfrac{1}{2}(1 - 0) = B$, $\underline{B = \tfrac{1}{2}}$

$\underline{x^*(t) = \tfrac{1}{2}(e^2 t + 1 - e^t)}$

By Mangasarian's Thm: $\underline{(x^*, u^*) \text{ is an optimal pair.}}$

(This solution may also be obtained from the Euler equation since x^* is C^2, and $\dot{x} = -u$ yields a problem in the Calculus of Variations)

12.2.2 / 9.2.2)

$\min_u \int_0^1 [x(t)+u(t)^2]\,dt$, $\dot{x}(t) = -u(t)$, $x(0)=0$, $x(1)$ free

(Remark: $\max H = -\min(-H)$, hence we can minimize H as a function of u for such min. problems)

$H(t,x,u,p) = x + u^2 + p(-u) = x + u^2 - pu$

which is a convex function of (x,u), by the 2nd der. test: $\frac{\partial^2 H}{\partial x^2} = 0 \geq 0$, $\frac{\partial^2 H}{\partial u^2} = 2 > 0$, $\frac{\partial^2 H}{\partial x \partial u} = 0$,

$\Delta_H = 0 \cdot 2 - 0^2 = 0 \geq 0$, so test yields convexity.

Hence Mangasarian's Thm. applies.

(a) $\frac{\partial H}{\partial u} = 2u - p = 0 \iff u = \frac{1}{2}p$; $\frac{\partial^2 H}{\partial u^2} = 2 > 0$ so it is a min.

By Mangasarian's Thm. $u^*(t) = \frac{1}{2}p(t)$ is an optimal control.

(b) $\frac{\partial H^*}{\partial x} = -\dot{p}(t)$, $\dot{p}(t) = -1$, $p(t) = -t + A$

Here $p(1) = 0$ since $x^*(1)$ is free (using (iii')). Hence

$A = +1$, $\underline{p(t) = t+1}$

Then $u^*(t) = \frac{1}{2}p(t) = \underline{\frac{1}{2}(t+1)}$

$\dot{x}^*(t) = -u^*(t) = -\frac{1}{2}(1+t)$, $x^*(t) = \frac{1}{2}(-t - \frac{1}{2}t^2) + B$

where $x^*(0) = B = \underline{0}$, $x^*(t) = \underline{-\frac{1}{2}t - \frac{1}{4}t^2}$

By Mangasarian's Thm. $\underline{(x^*, u^*) \text{ is an optimal pair}}$.

12.2.3 / 9.2.4

$$\max_u \int_0^{10} [1 - 4x(t) - 2u(t)^2]\,dt \quad , \quad \dot{x} = u \ , \ x(0) = 0 \ , \ x(10) \text{ free}$$

Here $H(t,x,u,p) = 1 - 4x - 2u^2 + pu$
which is concave of (x,u) for each $t \in [0,10]$, hence Mangasarian's Thm. applies. Suppose (x^*, u^*) is an optimal pair.

(a) $u = u^*(t)$ must maximize $H(t, x^*(t), u, p(t))$ for each $t \in [0,10]$.

Here $\dfrac{\partial H^*}{\partial u} = -4u + p = 0 \iff u = \frac{1}{4}p$. This is a max. since H was concave in (x,u), hence in u.

Hence $u^*(t) = \frac{1}{4}p(t)$, $t \in [0, 10]$

(b) $\dfrac{\partial H^*}{\partial x} = -\dot{p}(t)$, $-4 = -\dot{p}(t)$, $p(t) = 4t + A$

Here $p(10) = 0$ (since $x^*(10)$ is free), so $A = -40$

$p(t) = \underline{4t - 40}$, $u^*(t) = \frac{1}{4}p(t) = \underline{t - 10}$

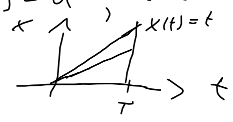
$\dot{x}^*(t) = u^*(t) = t - 10$, $x^*(t) = \frac{1}{2}t^2 - 10t + B$

$x^*(0) = 0 = B$. $x^*(t) = \underline{\frac{1}{2}t^2 - 10t}$

(If you solve this by Calculus of Variations, the Euler eq. becomes $\ddot{x} = 1$)

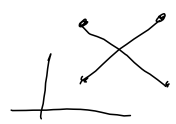
12.4.1 / 9.4.1

$$V(T) = \max \int_0^T x(t)\,dt \quad , \quad \dot{x} = u \ , \ x(0) = 0 \ , \ x(T) \text{ free}$$
$$u(t) \in [0,1] = \mathcal{U} \ , \ t \in [0,T]$$



(a) $x(t)$ represents a plane curve through origin since $x(0) = 0$
The slopes of $x(t)$ are between 0 and 1, since $\dot{x} = u$, $u(t) \in [0,1]$.
$\int_0^T x(t)\,dt$ gives the area under the graph of the curve, above the t-axis. It is geometrically clear that $x(t) = t$ gives the maximal area.

$$V(T) = \int_0^T t\,dt = \underline{\frac{T^2}{2}}$$



(b) Solution by Control Theory:

$H(t,x,u,p) = x + pu$, $u \in [0,1]$.
which is maximized for $u = 0$ or $u = 1$ (it is a straight line as a function of u!)

Hence $u^*(t) = \begin{cases} 1, & \text{if } p(t) > 0 \\ 0, & \text{if } p(t) < 0 \\ \text{undetermined} & \text{if } p(t) = 0 \end{cases}$

(b) Furthermore,

$\frac{\partial H^*}{\partial x} = -\dot{p} = 1$, $\dot{p}(t) = -1$, $p(t) = -t + A$

(iii)

$p(T) = 0$ as $x(T)$ is free

$-T + A = 0$, $\underline{A = T}$, $\underline{p(t) = T - t} \geq 0$, $t \in [0,T]$,

hence $\underline{u = u^*(t) = 1}$

$\dot{x}^*(t) = 1$, $x^*(t) = t + B$, $x^*(0) = 0 = B$

$\underline{x^*(t) = t}$

Here $H(t,x,u,p) = x + pu$ is linear of (x,u), hence is concave (and convex), so Mangasarian's Thm. says that $x^*(t) = t$, $u^*(t) = 1$, $t \in [0,T]$, is an optimal pair.

12.4.2 / 9.4.2

$\max \int_0^1 (1 - x^2 - u^2)\,dt$; $\dot{x} = u$, $x(0) = 0$, $x(1) \geq 1$, $u(t) \in \mathbb{R}$

We will assume this is a normal problem, so

$H(t,x,u,p) = 1 - x^2 - u^2 + pu$

which is concave of (x,u) (by the 2nd der. test)

Hence Mangasarian's applies.

(a) $u = u^*(t)$ must maximize H:

$\frac{\partial H}{\partial u} = -2u + p = 0$, $u = \frac{1}{2}p$, which yields a max as H is concave of u.

Hence $u = u^*(t) = \frac{1}{2}p(t)$, for any optimal control u^*.

(b) $\frac{\partial H^*}{\partial x} = -\dot{p}(t)$, $-2x^*(t) = -\dot{p}(t)$,

(*) $\dot{p} = 2x^*$ and (**) $\dot{x}^* = \frac{1}{2}p$ } so by (**) and (*) $\ddot{x}^* = \frac{1}{2}\dot{p} = x^*$, $\ddot{x}^* - x^* = 0$ ($r^2 - 1$ is the char. eq., $r = \pm 1$)

$x^*(t) = Ae^t + Be^{-t}$

$\dot{p}(t) = +2x^*(t) = +2(Ae^t + Be^{-t})$, $p(t) = +2Ae^t - 2Be^{-t} + C$

$x^*(0) = 0$, so $A + B = 0$, $B = -A$,

$x^*(t) = A(e^t - e^{-t})$, $p(t) = +2A(e^t - e^{-t}) + C$

$x^*(1) = A(e - e^{-1}) \geq 1$, so $\underline{A \geq \frac{1}{e - e^{-1}}}$

(iii): $p(1) \geq 0$ ($p(1) = 0$ if $x^*(1) > 0$)

From (**) $p(t) = 2\dot{x}^* = 2A(e^t + e^{-t}) = 2A(e^t \overset{?}{+} e^{-t}) + \boxed{C}$ $=0$

$\underline{p(t) = 2A(e^t + e^{-t})}$

If $x^*(1) = A(e^t - e^{-t}) > 1$, $A > \frac{1}{e - e^{-1}} > 0$

then $p(1) = 0 = 2A(e + e^{-1})$, $A = 0$! Contradiction

Hence

$x^*(1) = A(e - e^{-1}) = 1$, $A = \frac{1}{e - e^{-1}}$

$\left.\begin{array}{l} x^*(t) = \frac{1}{e - e^{-1}}\underline{(e^t - e^{-t})} \\ u^*(t) = \dot{x}^*(t) = \underline{\frac{e^t + e^{-t}}{e - e^{-1}}} \end{array}\right\}$ is an optimal pair.