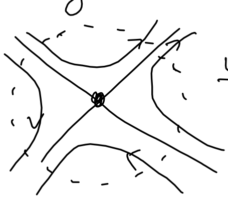
May 11. 2016 MA 2440

Consider the autonomous system

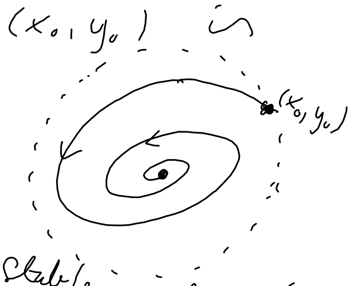
(1) $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, $t \geq t_0$

Suppose (x_*, y_*) is a critical point of (1), and let $(x(t), x(t))$ be a solution curve of (1) with $x(t_0) = x_0$, $y(t_0) = y_0$ (initial values)

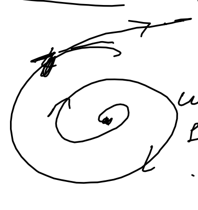
We say that (x_*, y_*) is <u>stable</u> if the points $(x(t), y(t))$ of the trajectory remain "close to" (x_*, y_*) whenever (x_0, y_0) is "sufficiently close to" (x_*, y_*)



<u>unstable</u>
(saddle point)



<u>Stable</u>
(spiral point)
<u>sink</u>



unstable
spiral point
<u>source</u>

<u>Definition</u>

(x_*, y_*) is <u>stable</u> if for all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|(x_*, y_*) - (x(t), y(t))\| < \varepsilon$$

whenever $\|(x_*, y_*) - (x_0, y_0)\| < \delta$

Otherwise (x_*, y_*) is called unstable.

<u>Def.</u> A critical point (x_*, y_*) of (1) is called a <u>node</u> if

Either 1) every trajectory approaches (x_*, y_*) as $t \to \infty$ (nodal sink)

or every trajectory recedes from (x_*, y_*) as $t \to \infty$ (nodal source)

OR ELSE: 2) every trajectory is tangent at (x_*, y_*) to some straight line through the point (x_*, y_*)

Proper node: Node that is not improper.

Improper node: There is a pair of "opposite" trajectories that are tangents at (x_*, y_*) to some straight line (different from the trajectory!) through (x_*, y_*)

$\left(\int_0^T (1 - t x_0)\,dt = \left[_0^T\, t - \tfrac{1}{2} t x_0 \right] = T^2 - \tfrac{1}{2} T x_0 = \text{maximum} \right)$

12.4.3 / 9.4.3 (a)

$$\max \int_0^T [1 - t x(t) - u(t)^2]\,dt, \quad \dot{x} = u, \; x(0) = x_0 > 0, \; x(T) \text{ free}$$
$$u(t) \in [0,1] \quad (t \in [0,T])$$

For each $t \in [0,T]$

$(x,u) \longmapsto H(t, x, u, p(t)) = 1 - tx - u^2 + p(t)u$

(this is a normal problem since $x(T)$ is free)

is concave by the 2nd-derivative test:

$$\frac{\partial^2 H}{\partial x^2} = 0 \le 0, \quad \frac{\partial^2 H}{\partial u^2} = -2 < 0, \quad \frac{\partial^2 H}{\partial x \partial u} = 0$$

$$\Delta_H = 0 \cdot (-2) - \left(\frac{\partial^2 H}{\partial x \partial u}\right)^2 = 0 - 0^2 = 0 \ge \underline{0}$$

Thus Mangasarian's Thm. applies. Suppose (x^*, u^*) is an optimal pair.

(A) $u = u^*(t)$ must maximize $H(t, x^*(t), u, p(t))$ for each $t \in [0,T]$.

$$\frac{\partial H^*}{\partial u} = -2u^*(t) + p(t) = 0 \iff u^*(t) = \frac{p(t)}{2} \text{ and } \underline{\frac{p(t)}{2} \in [0,1]}$$

Hence, if

$p(t) \in [0,2]$: This value, $u = u^*(t) = \frac{1}{2}p(t)$ yields a max. since $\frac{\partial^2 H}{\partial u^2} = -2 < 0$.

$p(t) \notin [0,2]$: $\frac{\partial H^*}{\partial u}$ never zero, so $u^*(t) = 0$ or $u^*(t) = 1$ will give the maximum (an endpoint of $[0,1]$!)

We will find $p(t)$:

$$\frac{\partial H^*}{\partial x} = -\dot{p}(t), \quad \text{that is} \quad -t = -\dot{p}(t), \quad \dot{p}(t) = t$$

$$p(t) = \tfrac{1}{2}t^2 + A, \quad p(T) = 0, \quad \underline{A = -\tfrac{1}{2}T^2}$$

$$p(t) = \underline{\tfrac{1}{2}(t^2 - T^2)} \qquad \text{Hence } p(t) \le 0, \; t \in [0,T], \text{ so}$$

$$p(t) \notin [0,2], \; t \ne T, \qquad p(T) = 0.$$

$H = 1 - tx - u^2 + p(t)u, \quad p(t) < 0$

is maximal for $u = u^*(t) = 0$ $(0 \le t < T)$ since
$1 - tx > -tx + p$ (since $p < 0 \le 1$)

Hence
$u^*(t) = 0$, $t \in [0, T)$
$u^*(T) = \frac{1}{2} p(T) = 0$ (since $x(T)$ is free)

Hence
$\dot{x}^*(t) = u^*(t) = 0$
$x^*(t) = \text{a constant} = x_0$, $t \in [0, T]$.

By Mangasarian's Thm. $(x^*, u^*) = (x_0, 0)$ is
an optimal pair (solving the max. problem)

(b) Same as (a), but $u(t) \in [-1,1]$, $T > 2$:

$\max \int_0^T (1 - tx - u^2)\,dt$, $\dot{x} = u$, $x(0) = x_0$, $x(T)$ free, $u(t) \in [-1,1]$, $t \in [0,T]$.

Calculations as in (a) and

$$p(t) = \tfrac{1}{2}(t^2 - T^2),$$

and $\frac{\partial H}{\partial u} = -2u + p = 0 \iff p = 2u$, iff $p \in [-2,2]$

$\frac{\partial^2 H}{\partial u^2} = -2 < 0$

Here $0 \geq p(t) \geq -\frac{1}{2}T^2$, where $T^2 > 4$, so $-\frac{1}{2}T^2 < -2$

Hence

(1) For $p(t) \in [-2, 0]$: $u = u^*(t) = \frac{1}{2}p(t) = \frac{1}{4}(t^2 - T^2)$

(2) For $p(t) \in [-\frac{1}{2}T^2, -2)$: Max of H occurs at an endpoint $u = u^*(t) = -1$ or 1.

Now

$H(t, x^*(t), -1, p(t)) = 1 - t x^*(t) - 1 - p(t)$

$\geq H(t, x^*(t), 1, p(t)) = 1 - t x^*(t) - 1 + p(t)$ (since $p(t) \leq 0$)

Hence $u^*(t) = -1$ if $p(t) \in [-\frac{1}{2}T^2, -2)$

Next, $p(t) \in [-2, 0] \iff \frac{1}{2}(t^2 - T^2) \geq -2 \iff t^2 - T^2 \geq -4$

$\iff t^2 \geq T^2 - 4 \iff t \geq \sqrt{T^2 - 4}$ (as $t \geq 0$)

Moreover,

$\frac{1}{2}(t^2 - T^2) = p(t) \geq -\frac{1}{2}T^2$

so $p(t) \in [-\frac{1}{2}T^2, -2) \iff \frac{1}{2}(t^2 - T^2) < -2$

$\iff t < \sqrt{T^2 - 4}$

Hence
$$u^*(t) = \begin{cases} \frac{1}{4}(t^2 - T^2), & \text{if } T \geq t \geq \sqrt{T^2-4} \\ -1, & \text{if } 0 \leq t < \sqrt{T^2-4} \end{cases}$$

$$x^*(t) = \begin{cases} \frac{1}{12}t^3 - \frac{1}{4}T^2 t + \boxed{C}, & t \in [\sqrt{T^2-4}, T] \\ -t + x_0, & t \in [0, \sqrt{T^2-4}) \end{cases}$$

x^* is continuous, hence at $t = \sqrt{T^2-4}$:

$x^*(\sqrt{T^2-4}) = \frac{1}{12}(T^2-4)\sqrt{T^2-4} - \frac{1}{4}T^2\sqrt{T^2-4} + C$

$= \lim_{t \to \sqrt{T^2-4}^-} (-t + x_0) = -\sqrt{T^2-4} + x_0$

$C = -\frac{1}{12}(T^2-4)\sqrt{T^2-4} + \underbrace{\frac{1}{4}T^2\sqrt{T^2-4} - \sqrt{T^2-4}} + x_0$

$= -\frac{1}{12}(T^2-4)\sqrt{T^2-4} + \frac{1}{4}(T^2-4)\sqrt{T^2-4} + x_0$

$= \frac{1}{6}(T^2-4)\sqrt{T^2-4} + x_0$

By Mangasarian's Thm. (x^*, u^*) solves the Max. problem.

EP 7.2(9)

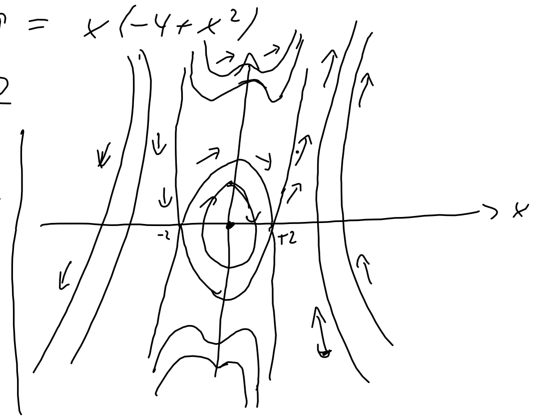
$x'' + 4x - x^3 = 0$, $x'' = -4x + x^3 = F(x, x')$

Equilibrium solutions for $x(t) = x_0$ = a constant.

Then $x'' = 0$ (and $x' = 0$),

hence $F(x, x') = -4x + x^3 = 0 = x(-4 + x^2)$

$x(t) = 0$ or $x(t) = \pm 2$



An equivalent 1-st order system is:

$$(*)\begin{cases} x' = y = f(x,y) \\ y' = -4x + x^3 = g(x,y) \end{cases}$$

On matrix form

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ x^3 \end{bmatrix}$$

Here $\det \begin{vmatrix} 0 & 1 \\ -4 & 0 \end{vmatrix} = 4 \neq 0$

so system is almost linear at (0,0)

(Only 3 critical points, so they are all isolated)

At $x = 2$, $y = 0$: $f(x,y) = y$, $g(x,y) = -4x + x^3$

$\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 1$, $\frac{\partial g(2,0)}{\partial x} = (-4 + 3x^2)_{(2,0)} = -4 + 12 = 8$, $\frac{\partial g}{\partial y} = 0$

So $J(2,0) = A = \begin{bmatrix} 0 & 1 \\ 8 & 0 \end{bmatrix}$ Eigenvalues: $\lambda^2 - 8 = 0$

$\lambda = \pm 2\sqrt{2}$: Real, opposite signs

Hence (2, 0) is a saddle point (unstable)

At $x = -2, y = 0$: Same A as before

(Unstable) saddle point

Cfr. Thm. 2 p. 508, in particular part 3