MAT 2440. May 18, 2016.

Autonomous systems, almost linear at critical point (x_0, y_0)

$$\frac{dx}{dt} = f(x,y) \quad , \quad \frac{dy}{dt} = g(x,y)$$

Complex eigenvalues of $\boxed{A = J(x_0, y_0)}$:

$\lambda = p + qi$, $\bar{\lambda} = p - qi$ $(q \neq 0,\ p, q \text{ real})$

with associated complex eigenvectors:

$\vec{v} = \vec{a} + i\vec{b}$, $\bar{\vec{v}} = \vec{a} - i\vec{b}$

Have complex solutions of linearized system: $\dot{\vec{X}} = A\vec{X}$

$$\vec{X}(t) = e^{pt}(\cos qt + i \sin qt)(\vec{a} + i\vec{b})$$
$$= e^{pt}(\cos qt\, \vec{a} - \sin qt\, \vec{b} + i(\sin qt\, \vec{a} + \cos qt\, \vec{b}))$$

This gives 2 linearly indep. real solutions:

$$(\mathrm{I}) \begin{cases} \vec{X}_1(t) = e^{pt}(\cos qt\, \vec{a} - \sin qt\, \vec{b}) \\ \vec{X}_2(t) = e^{pt}(\sin qt\, \vec{a} + \cos qt\, \vec{b}) \end{cases}$$

(corresp. to $\vec{v} = \vec{a} + i\vec{b}$)

Purely imaginary λ : $p = 0$, $\lambda = qi$

$$(\mathrm{II}) \begin{cases} \vec{X}_1(t) = \cos qt\, \vec{a} - \sin qt\, \vec{b} \\ \vec{X}_2(t) = \sin qt\, \vec{a} + \cos qt\, \vec{b} \end{cases}$$

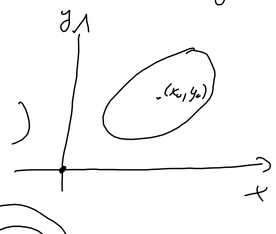
Solutions are periodic of period $\frac{2\pi}{q}$

$k = 0, \pm 1, \pm 2, \ldots$

$$\left(e^{iqt} = e^{i(qt + 2k\pi)} = e^{iq(t + \frac{2k}{\pi})} \right)$$

Hence (x_0, y_0) is a center

Can be shown that solutions form ellipses centered at (x_0, y_0)



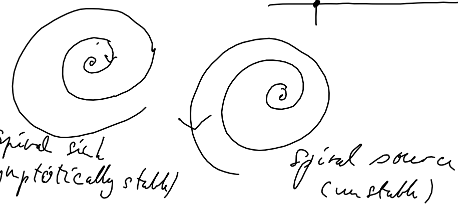
(Sufficient to show solutions satisfy an eq.

$$\alpha x^2 + \beta y^2 + \gamma xy + \delta x + \varepsilon y + \mu = 0 \;)$$

$p \neq 0$, not purely imaginary:

Spiral sink if $p < 0$

Spiral source if $p > 0$



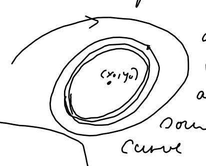
Concerning the nonlinear system: λ_1, λ_2 eigenvalues.

(cf. p. 504 E p)

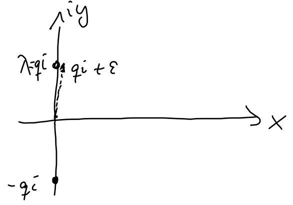
2. λ_1, λ_2 purely imaginary:

Then (x_0, y_0) is either a center or a spiral point

(stable, unstable, asymptotically stable



asymptotically stable, approaching some closed curve as $t \to \infty$



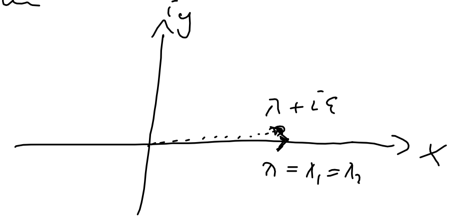
To show it is a center, it suffices to show the curves are periodic

See also: Exam 2011 (Problem 1)

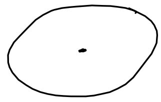
1. (of Thm. 2 p. 508)

$\lambda_1 = \lambda_2$: real, equal eigenvalues :

critical point of nonlinear system is either node or a spiral point



3. In the other cases we have the same type of critical point for both systems (linearized and nonlinear).



Example. $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -2x - 2y$ | a linear system

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Eigenvalues : $0 = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+2 \end{vmatrix} = \lambda(\lambda+2) + 2 = \lambda^2 + 2\lambda + 2 = 0$

$\lambda = \frac{1}{2}\left[-2 \pm \sqrt{4-8}\right] = -1 \pm i$

A spiral sink.

Example. (p. 493)

$$(*) \begin{cases} x(t) = A\cos\omega t + B\sin\omega t \\ y(t) = -A\omega\sin\omega t + B\omega\cos\omega t \end{cases}$$

$\omega > 0$

are solutions of $\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x \end{cases}$ (0,0) is critical

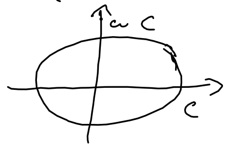
$A = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}$ eigenvalues $\lambda = \pm\omega i$

Eigenvectors $\begin{bmatrix} 1 \\ i\omega \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ \omega \end{bmatrix}$ Calculations yield (*)

In (*) :

$C = \sqrt{A^2+B^2}$, $\frac{A}{C} = \cos\alpha$, $\frac{B}{C} = \sin\alpha$ $\left(\frac{A^2}{C^2} + \frac{B^2}{C^2} = 1\right)$

$$\left.\begin{aligned} x(t) &= C\cos(\omega t - \alpha) \\ y(t) &= C\omega\sin(\omega t - \alpha) \end{aligned}\right\} \quad \frac{x(t)^2}{C^2} + \frac{y(t)^2}{\omega^2 C^2} = 1$$



Ellipses centered at (0, 0)

Exam June 2006. Problem 2.

(1) $\max \int_{-1}^{1} (tx - u^2)\,dt$, $\dot{x} = x + u^2$, $u(t) \in [0,1]$.
$x(-1) = -2e^{-1} - 1$, $x(1)$ free

(a) Suppose (x^*, y^*) is an optimal pair.
The Hamiltonian:

$$H(t,x,u,p) = tx - u^2 + p(x + u^2)$$
$$= (t+p)x + (p-1)u^2$$

According to the Maximum Principle, there is a continuous, piecewise C^1-function p, such that for each $t \in [-1,1]$, $u = u^*(t)$ maximizes
$$H(t, x^*(t), u, p(t))$$

Here $\frac{\partial H}{\partial u} = 2u(p-1) = 0 \iff u = 0 \quad (\text{if } p \neq 1)$
$\frac{\partial^2 H}{\partial u^2} = 2(p-1) < 0 \iff p < 1$

Hence $u = u^*(t) = 0$ if $p(t) < 1$

If $p(t) = 1$, then H is constant at u, so every $u \in [0,1]$ maximizes H.

If $p(t) > 1$, then $\frac{\partial^2 H}{\partial u^2} > 0$, $u = 0$ minimizes H.
The maximum of H is then attained at $u = 1$ (the other endpoint of $[0,1]$). Hence $u^*(t) = 1$

$$u^*(t) = \begin{cases} 0, & \text{if } p(t) < 1 \\ 1, & \text{if } p(t) > 1 \\ \text{undetermined} & \text{if } p(t) = 1 \end{cases}$$ (we may let $u^* = 0$ in this case)

By the Max. Principle again:

$$-\dot{p}(t) = \frac{\partial H^*}{\partial x} = t + p \quad ; \quad \dot{p} + p = -t$$

$$p(t) = Ae^{-t} + Bt + C$$

Then $\dot{p}(t) + p(t) = -Ae^{-t} + B + Ae^{-t} + Bt + C$
$= B + C + Bt \equiv -t$

$B = -1$, $C + B = 0$; $B = -1, \ C = 1$

$$p(t) = Ae^{-t} - t + 1$$

Since $x(1)$ is free, $p(1) = 0$:

$$0 = Ae^{-1} - 1 + 1 = Ae^{-1} , \quad A = 0$$

$p(t) = 1 - t$: So $p(t) < 1 \iff 1 \geq t > 0$

Hence

$$u^*(t) = \begin{cases} 0, & \text{if } 0 \le t \le 1 \\ 1, & \text{if } -1 \le t < 0 \end{cases}$$

$t \in [0,1]$:

Then $\dot{x}^* = x^*$, $\quad x^*(t) = k e^t$

$t \in [-1, 0)$: $\quad u^* = 1$

$\dot{x}^* - x = 1$, $\quad x^*(t) = B e^t - 1$

Here $x^*(-1) = \underbrace{-2e^{-1} - 1 = B e^{-1} - 1}$, $\quad \underline{B = -2}$

$\underline{x^*(t) = -2e^t - 1}$

x^* is continuous at $t = 0$, hence

$$x^*(0) = \lim_{t \to 0^+} x^*(t) = k$$

$$= \lim_{t \to 0^-} x^*(t) = -2e^0 - 1 = -3, \quad \underline{k = -3}$$

$$x^*(t) = \begin{cases} -2e^t - 1, & t \in [-1, 0) \\ -3e^t, & t \in [0, 1] \end{cases}$$

$\underline{(x^*, u^*)}$ is the <u>only solution candidate</u>

(b) Here H is concave in $(x, u) \iff$

$p(t) \le 1 \iff \underline{t \in [0, 1]}$

Mangasarian's Thm. does <u>not</u> apply.

Here

$$\hat{H}(t, x, p(t)) = \max_{u \in [0,1]} H(t, x, u, p(t))$$

$$= \max_{u \in [0,1]} [\,(t + p(t))x + (p(t) - 1)u^2\,]$$

$$= \begin{cases} (t + p(t))x, & 0 \le t \le 1 \\ (t + p(t))x + p(t) - 1, & -1 \le t < 0 \end{cases}$$

which is linear in x, hence is concave in x.

By Arrow's Thm, (x^*, u^*) is really an optimal <u>pair for the problem.</u>