MAT 2440. 24. May, 16. SET 15

Exam 2011

1. $\frac{dx}{dt} = 4y^3 - 4xy = f(x,y)$

$\frac{dy}{dt} = 2y^2 - 2x^3 = g(x,y)$

(a) Critical points: $f(x,y) = 0$ & $g(x,y) = 0$,

$y(y^2 - x) = 0$ & $y^2 = x^3$

(i) $y = 0$: $(0, 0)$

(ii) $y^2 = x$ $(y \neq 0)$: $x = x^3$; $x = 0$ or $x^2 = 1$
$(x = 0)$ or $x = \pm 1$
$y \neq 0$

$x = \pm 1$ yield $y^2 = x \geq 0$; so $x = 1$, $y = \pm 1$

$(1, 1)$, $(1, -1)$

There are exactly 3 critical points: $(0,0)$, $(1, \pm 1)$

We linearize at $(1, 1)$: The Jacobian matrix:

$$J(1,1) = \begin{bmatrix} -4y & 12y^2 - 4x \\ -6x^2 & 4y \end{bmatrix}_{(1,1)} = \begin{bmatrix} -4 & 8 \\ -6 & 4 \end{bmatrix} = A$$

If $u = x - 1$, $v = y - 1$, $(u,v) = (0,0)$ is a critical point

Get

$$\left.\begin{aligned} \frac{du}{dt} &= -4u + 8v \\ \frac{dv}{dt} &= -6u + 8v \end{aligned}\right\} ,$$

which is the linearized system in (u,v) coordinates

Since $(1,1) = (x,y)$ is isolated (only finitely many crit. pts.)

and $|J(1,1)| \neq 0$, the system is almost linear

at $(x,y) = (1,1)$.

Eigenvalues of A:

$$0 = \begin{vmatrix} -4-\lambda & 8 \\ -6 & 4-\lambda \end{vmatrix} = (\lambda-4)(\lambda+4)+48 = \lambda^2+32$$

$\lambda = \pm i 4\sqrt{2}$, purely imaginary.

Hence the linearized system has a stable center at $(x,y)=(1,1)$.

For the nonlinear system the point $(1,1)$ is either a center or a spiral point; stable, asymptotically stable, or unstable. (cfr. Thm. 2 Sec. 7.3 of EP)

(b) (*) $\dfrac{dy}{dx} = \dfrac{2y^2-2x^3}{4y^3-4xy}$ $\left(= \dfrac{dy}{dt} \Big/ \dfrac{dx}{dt} \right)$ $(y \neq 0, y^2 \neq x)$

gives an exact differential form

Let $P(x,y) = (2y^2-2x^3)$, $Q(x,y) = -(4y^3-4xy)$

Then (*) can be written

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

Here

$$\frac{\partial P}{\partial y} = 4y = \frac{\partial Q}{\partial x} = 4y$$

Hence the form is exact.

Thus there is a "potential" $\varphi(x,y)$ such that

$\dfrac{\partial \varphi}{\partial x} = P$, $\dfrac{\partial \varphi}{\partial y} = Q$; $\varphi(x,y) = C$ solves the eq.

Hence

$$\varphi(x,y) = \int P(x,y)\,dx + A(y) = \int (2y^2-2x^3)\,dx + A(y)$$

$$= 2xy^2 - \tfrac{1}{2}x^4 + A(y)$$

and $\dfrac{\partial \varphi}{\partial y} = \underline{4xy} + A'(y) = Q(x,y) = \underline{4xy} - 4y^3$

$A'(y) = -4y^3$, $A(y) = -y^4 + C_1$ (let $C_1 = 0$)

$$\varphi(x,y) = 2xy^2 - \tfrac{1}{2}x^4 - y^4$$

The solutions are given implicitly by

$$\underline{-2xy^2 + \tfrac{1}{2}x^4 + y^4 = C}$$

This can be written (since $(x-y^2)^2 = x^2 + \underline{y^4} - \underline{2xy^2}$)

$(x-y^2)^2 - x^2 + \tfrac{1}{2}x^4 = C_1$

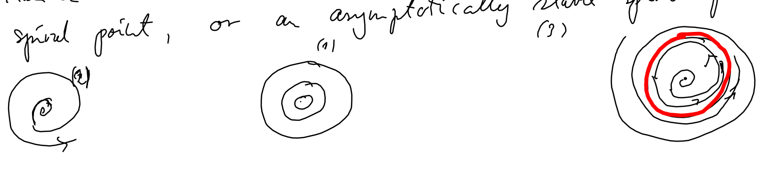
$(x-y^2)^2 - \tfrac{1}{2}(x^4-2x+1) + \tfrac{1}{2} = C_1$; (**) $\underline{(x-y^2)^2 + \tfrac{1}{2}(x^2-1) + C = 0}$

$\underline{(1,1) \text{ is no sink:}}$ Suppose $(x(t), y(t))$ is a sol. curve starting close to, but not at $(1,1)$ (at (x_0,y_0)), then $\underline{C \neq 0}$.

If it were a sink, then $\lim\limits_{t\to\infty} x(t) = y(t) = 1$

Hence by (**) $0 + 0 = -C$, $\underline{C = 0}$, a $\underline{\text{contradiction}}$.

Hence $(1,1)$ is either a (1) center, or an (2) unstable spiral point, or an asymptotically stable spiral point (which is no sink) (3)



Exam 2011 3.

(1) $\max \int_0^{\pi} (x^2 - u^2)\,dt$, $x(0) = 1$, $x(\pi)$ is free, $\dot{x} = u$, $u(t) \in [0,1]$ $(t \in [0,\pi])$.

(a) Let $H(t, x, u, p) = x^2 - u^2 + pu$. Assume that (x^*, u^*) is an optimal pair for (1). According to the Max. Principle there is a continuous and piecewise continuously differentiable (piecewise C^1) function p, such that for all $t \in [0, \pi]$

(2) $u = u^*(t)$ maximizes $H(t, x^*(t), u, p(t))$

and

(3) $\frac{\partial H}{\partial x}(t, x^*(t), u^*(t), p(t)) = -\dot{p}(t)$ (written $\frac{\partial H^*}{\partial x} = -\dot{p}(t)$)

except at the discontinuities of u^*.

Further, $x(\pi)$ is free $\Rightarrow$ $p(\pi) = 0$

If $u = u^*(t)$, $x = x^*(t)$, then (3) says

$\dot{p}(t) = -2x(t)$,

where $\dot{x} = u \geq 0$, so x is increasing.

Since $x(0) = 1 > 0$, we have $x(t) > 0$ on $[0, \pi]$. Hence $-2x < 0$, so that $\dot{p}(t) = -2x(t) < 0$, hence <u>p is decreasing</u> (strictly)

(b) Next $\frac{\partial H}{\partial u} = -2u + p = 0 \iff u = \frac{1}{2}p$

where $\frac{\partial^2 H}{\partial u^2} = -2 < 0$. Hence $u = u^*(t) = \frac{1}{2}p(t)$ maximizes

H <u>if</u> $u = \frac{1}{2}p(t) \in [0, 1]$, <u>$p(t) \in [0, 2]$</u>

We must discuss:

<u>$p(t) > 2$</u>: Then $\frac{\partial H}{\partial u}$ is never 0. Hence, since H is continuous in u, the max. is attained at an endpoint $u = 0$ or $u = 1$.

Now
$H(t, x, 0, p) = x^2 < H(t, x, 1, p) = x^2 - 1 + p \quad (p - 1 > 0)$

Hence $\underline{u = u^*(t) = 1}$.

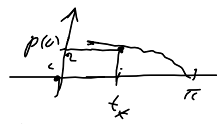
the endpoints must be considered.

$\underline{p(t) < 0:}$ Again, $H(t, x, 0, p) = x^2 > H(t, x, 1, p) = x^2 - 1 + p \quad (p-1<0)$

Hence $\underline{u = u^*(t) = 0}$

This means
$$u^*(t) = \begin{cases} 0, & p(t) < 0 \\ \frac{1}{2}p(t), & p(t) \in [0, 2] \\ 1 & p(t) > 2 \end{cases}$$

(c) Since p is strictly decreasing and $p(\pi) = 0$, we have



$p(t) > 0$ on $[0, \pi)$.

Hence there is a $t_* \in [0, \pi)$ such that $p(t) \in [0, 2]$ for $t \in [t_*, \pi]$ (t_* could perhaps be 0).
(we used that p is continuous.)

Let $\bar{x} = x^*(\pi)$.

$\underline{\text{On } [t_*, \pi]}$ we find:

$u = u^*(t) = \frac{1}{2}p(t)$, $\underline{\dot{x} = u}$

Hence

(i) $\dot{p}(t) = -2x^*(t)$

(ii) $\dot{x}^*(t) = \frac{1}{2}p(t)$

Thus $\ddot{p} = -2\dot{x}^* = -p$, $\quad \ddot{p} + p = 0$

$p(t) = A\cos t + B\sin t$, $\quad p(\pi) = 0 = -A$,

$p(t) = B\sin t$

$x^*(t) \overset{(i)}{=} -\frac{1}{2}\dot{p}(t) = -\frac{1}{2}B\cos t$, $\quad x^*(\pi) = \bar{x} = +\frac{1}{2}B$

$\underline{x^*(t) = -\bar{x}\cos t, \quad p(t) = 2\bar{x}\sin t}$ $\quad (u^*(t) = \frac{1}{2}p(t) = \bar{x}\sin t)$

(d) We cannot have $t_* = 0$:

If $t_* = 0$, then $p(t) = 2\bar{x}\sin t$ on $[0, \pi]$
which is not decreasing, a $\underline{\text{contradiction}}$.

Define t_* by $p(t_*) = 2 = 2\bar{x}\sin t_*$, $\quad \underline{\bar{x} = \frac{1}{\sin t_*}}$

$\underline{t \in [0, t_*]}$: $\quad \dot{x}^*(t) = u^*(t) = 1$

$x^*(t) = t + a \quad , \quad x^*(0) = a = 1$

$x^*(t) = \underline{t+1}$

$\dot{p}(t) = -2x^*(t) = -2(t+1) \quad ; \quad \boxed{p(t) = -t^2 - 2t + b}$

$p(0) = \bar{p} = \underline{b} \qquad \underline{p(t) = -t^2 - 2t + \bar{p}}$

Since x^* is continuous at t_*, we have

$$x^*(t_*) = t_* + 1 = \lim_{t \to t_*^+} -\bar{x}\cos t = -\bar{x}\cos t_*$$

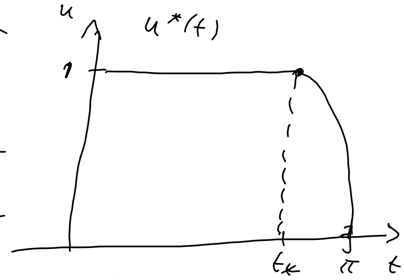
$$\overset{(d)}{=} -\frac{1}{\sin(t_*)}\cos t_* = \underline{-\frac{1}{\tan(t_*)}}$$

Furthermore, $p(t_*) = -t_*^2 - 2t_* + \bar{p} = 2\bar{x}\sin t_*$

as p is continuous at t_*. Hence

$$-t_*(t_* + 1) + \bar{p} = 2$$

$$\bar{p} = 2 - t_* \cdot \frac{1}{\tan t_*} \quad , \quad \underline{\bar{x} = \frac{1}{\sin t_*}}$$



Remark: Neither Mangasarian's nor Arrow's Theorems apply here!