MAT 2440 SET 16, May 25, 2016

Exam 2013 2.

$$(1)\quad \begin{cases} \dot{x} = -y + xy^2 & = f(x,y) \\ \dot{y} = 4x - 4x^2y & = g(x,y) \end{cases}$$

Critical points when $f(x,y) = g(x,y) = 0$. That is

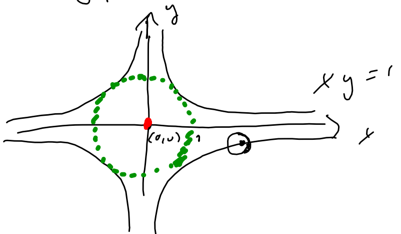
$$y(xy-1) = 0 \quad \& \quad 4x(1-xy) = 0$$

$(0,0)$ is a critical point.

Consider $y \neq 0$, $xy = 1$, then 2nd eq. gives nothing more.

Hence all points $(x, \frac{1}{x})$, $x \neq 0$

on the hyperbola $xy = 1$ are crit. points.



(b) $(0,0)$ is clearly isolated as $D = \{(x,y) : x^2+y^2 < 1\}$ contains no other critical points.

No point on $xy=1$ is an isolated crit. point as any open disk around such a point contains infinitely many other points on $xy = 1$.

The system (1) is almost linear at $(0,0)$:

Furthermore, $$J(0,0) = \begin{bmatrix} y^2 & -1+2xy \\ 4-8xy & -4x^2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} = A$$

So $|J(0,0)| = 4 \neq 0$.

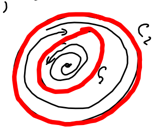
Clearly f and g are C^1-functions (polynomials), (hence the error terms of the 1st order Taylor formula go to zero faster than $\frac{1}{\sqrt{x^2+y^2}}$).

Consequently, the system is almost linear at $(0,0)$.

The type: $$|\lambda I - A| = \begin{vmatrix} \lambda & 1 \\ -4 & \lambda \end{vmatrix} = \lambda^2 + 4 = 0$$

$\Leftrightarrow$ $\lambda = \pm 2i$, hence eigenvalues are imaginary

Hence the linearized system has a center (stable) at $(0,0)$. However, the nonlinear system is either a center or a spiral point. Can be stable, unstable, or asymptotically stable.



(c) We solve (1):

Alternative 1 : $\frac{\dot y}{\dot x} = \frac{dy}{dx} = \frac{4x-4x^2y}{-y+xy^2}$ gives an exact differential form (as in Exam 2011).

Alternative 2 Multiply $\underbrace{}_{}$ the eq. of (1) by $4x$ and y, respectively

$$4x\dot x = -4xy + 4x^2y^2$$
$$y\dot y = 4xy - 4x^2y^2$$

We add the equations:

$$4x\dot x + y\dot y = 0,$$

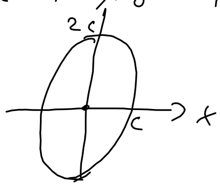
$$\frac{d}{dt}\left(2x^2 + \frac{1}{2}y^2\right) = 0 \quad . \quad \text{Integration then yields:}$$

$$2x^2 + \frac{1}{2}y^2 = k \qquad (k \geq 0)$$

$$x^2 + \frac{1}{4}y^2 = k_1 \quad , \quad \text{or} \quad \frac{x^2}{C^2} + \frac{y^2}{(2C)^2} = 1 \; , \; C > 0$$

$(0,0)$ if $C=0$

Ellipses centered at $(0,0)$



In particular, we conclude that $(0,0)$ is a center for (1)

Problem 3. (a) $\max_u \int_0^2 (x-u)\,dt$, $\dot x = x+u$, $x(0)=0$, $x(2)$ free, $u(t)\in[0,1]$, $t\in[0,2]$.

The Hamiltononian of this normal problem is (problem is normal since $x(2)$ is free):

$$H(t,x,u,p) = x - u + p(x+u) = x(1+p) + u(p-1)$$

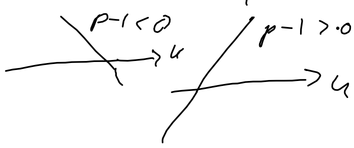
This is a linear function (1st order) of (x,u), hence is concave (and convex) in (x,u).

Hence if (x^*,u^*) is an admissible pair satisfying the conditions of the Maximum Principle, (x^*,u^*) will be optimal by Mangasarian's Theorem.

By the Max. Principle, there must exist a piecewise C^1-function p (which is continuous) such that

$u = u^*(t)$ maximizes $H(t, x^*(t), u, p(t))$ for each fixed $t \in [0,2]$.

Since H was linear, the max. must be attained at an endpoint of $[0,1]$ (H is continuous). Hence



$$u = u^*(t) = \begin{cases} 1 & \text{if } p(t) > 1 \quad : \ t \in [0, 2-\ln 2] \\ 0 & \text{if } p(t) < 1 \quad : \ t \in (2-\ln 2, 2] \\ \text{undetermined} & \text{if } p(t) = 1 \quad : \ t = 2-\ln 2 \end{cases}$$

Furthermore, $\frac{\partial H^*}{\partial x} = -\dot{p}(t)$

Hence $p + 1 = -\dot{p}$, $\quad \dot{p} + p = -1$, $\quad$ since $\underline{x(2) \text{ is free}}$

$p(t) = Ae^{-t} - 1$, where $p(2) = Ae^{-2} - 1 = 0$, $\underline{A = e^2}$

$p(t) = e^{2-t} - 1$ if $p(t) > 1$

Now $e^{2-t} - 1 > 1 \iff e^{2-t} > 2 \iff 2 - t > \ln 2$

$\iff \underline{t < 2 - \ln 2} \iff t \in [0, 2-\ln 2)$

From $\dot{x} - x = u$, so

$\dot{x} - x = 0$, if $p(t) < 1$

$x^*(t) = Be^t$, $t \in (2-\ln 2, 2]$

Since x^* is continuous, we have

$$\lim_{t \to 2-\ln 2^-} x^*(t) = \lim_{2-\ln 2^+} x^*(t) = Be^{2-\ln 2} = B\tfrac{1}{2}e^2$$

In addition:

$\dot{x} - x = 1$ if $t \in [0, 2-\ln 2)$

$x^*(t) = Ce^t - 1$ —"—

$x^*(0) = 0 = C - 1$, $\underline{C = 1}$ $\quad x^*(t) = e^t - 1$ on $[0, 2-\ln 2]$

The limit cond. at $2 - \ln 2$ yields:

$\frac{B}{2}e^2 = e^{2-\ln 2} - 1 = \frac{1}{2}e^2 - 1$, $\quad B = \frac{2}{e^2}(\frac{1}{2}e^2 - 1) = \underline{1 - \frac{2}{e^2}}$

Hence
$$x^*(t) = \begin{cases} (1 - 2e^{-2})e^t, & t \in (2-\ln 2, 2]. \\ e^t - 1, & t \in [0, 2-\ln 2] \end{cases}$$

By Mangasarian's Theorem $\underline{(x^*, u^*) \text{ is an optimal pair}}$ (as remarked previously).

(b) This is the same as in part (a), but with the new endpoint condition: $\underline{x(2) = \frac{1}{2}e^2}$

We find the same conditions on u^* and the same differential equations of x^* and p as before.

First, we cannot have $p(t) < 1$ on all of $(0, 2)$, since this would imply that $x^*(t) = Ce^t$ for all $t \in [0,2]$, which contradicts the endpoint conditions
$x^*(0) = 0$ and $x^*(2) = \frac{1}{2}e^2$

A similar argument shows that the inequality $p(t) > 1$ does not hold on all of $(0, 2)$.
Hence, as p is continuous, $p(t) - 1$ attains both positive and negative values on $(0, 2)$. Then there is a point s in $(0,2)$ such that $p(s) - 1 = 0$, $p(s) = 1$.
Then $p(s) = ae^{-s} - 1 = 1$, $a = 2e^s$, so
$$p(t) = 2e^s e^{-t} - 1, \quad t \in [0,2].$$
Now $2e^t$ is strictly increasing. Hence,
$$t \in (s, 2] \Rightarrow 2e^s e^{-t} < 2^t e^{-t} = 2 \Rightarrow p(t) < 1$$
and $t \in [0, s) \Rightarrow 2e^s e^{-t} > 2e^t e^{-t} = 2 \Rightarrow p(t) > 1$

We have shown:
$p(t) > 1$ on $[0, s)$
$p(t) < 1$ on $(s, 2]$

Hence, by the endpoint condition
$x^*(0) = b - 1 = 0$, $b = 1$,
$x^*(2) = Ce^2 = \frac{1}{2}e^2$, $C = \frac{1}{2}$

Now x^* is continuous at $t = s$, so
$$x^*(s) = e^s - 1 = \tfrac{1}{2}e^s \Rightarrow s = \ln 2$$
Thus $p(t) = 4e^{-t} - 1$ (as $p(\ln 2) = 1$ and p is strictly decreasing)

and $u^*(t) = 1$, $x^*(t) = e^t - 1$, $t \in [0, \ln 2]$
$u^*(t) = 0$, $x^*(t) = \frac{1}{2}e^t$, $t \in (\ln 2, 2]$

As in (a), Mangasarian's Thm. implies that this is an optimal pair.

Examples of: Centers for linearized systems that are no longer centers for the nonlinear system