We must elect a student representative today (or next week): Try to decide this during the break.

Problem Set 4. (Feb. 2. 2016)

2.2 (25) $Ly = y'' + py' + q$

Let y_1, y_2 satisfy

$L(y_1) = f(x)$, $L(y_2) = g(x)$

Claim: $L(y_1 + y_2) = f(x) + g(x)$

Proof. Let $y = y_1 + y_2$.

Then

$L(y) = y'' + py' + qy = (y_1'' + y_2'') + p(y_1' + y_2') + q(y_1 + y_2)$

$= (y_1'' + py_1' + qy_1) + (y_2'' + py_2' + qy_2) = L(y_1) + L(y_2)$

$= f(x) + g(x)$.

This proves the claim.

26/(a)

(i) $y'' + 2y = 4$ and (ii) $y'' + 2y = 6x$

Then (i) has a particular solution $y_p(x) = c$ (for all x)

Thus $y_p'' + 2y_p = 0 + 2c = 4$, $\underline{c = 2}$

(ii) has a particular solution $y_p(x) = Ax + B$:

$L(y_p) = 0 + 2(Ax+B) \equiv 6x$

Hence $2A = 6$, $B = 0$; $A = 3$, $B = 0$

So $y_p(x) = \underline{3x}$

(b) By (25) a partic. sol. of $y'' + 2y = 6x + 4$: $\underline{y_p(x) = 3x+2}$

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2.2 (36) Let

(*) $L(y) = y'' + p(x)y' + q(x)y = 0$

Suppose we know a solution $y_1(x)$ (on an interval I on which p and q are continuous)

Reduction of order: In order to find a solution y_2, independent of y_1, we substitute

$y_2(x) = v(x)\, y_1(x)$, $x \in I$.

into (*):

$y_2' = v' y_1 + v y_1'$

$y_2'' = v'' y_1 + 2v' y_1' + v y_1''$

Hence

$L(y_2) = y_2'' + p y_2' + q y_2$

$= (v'' y_1 + 2v' y_1' + v y_1'') + p(v' y_1 + v y_1') + q v y_1$

$= v(y_1'' + p y_1' + q y_1) + y_1(v'' + pv') + 2v' y_1'$

$= 0 + y_1 v'' + (y_1 p + 2y_1')v'$

Hence $L(y_2) = 0 \iff y_1 v'' + (y_1 p + 2y_1')v' = 0$

$u = v'$ $\iff$ $y_1 u' + (py_1 + 2y_1')u = 0$

$2\int \frac{y_1'}{y_1}dx$
$= 2\ln|y_1| + K$
$= \ln(y_1)^2 + K$
Let $K = 0$

Integrating factor of

$u' + \frac{py_1 + 2y_1'}{y_1} u = 0$ (if y_1 is never 0 on I)

is $\rho(x) = e^{\int \frac{py_1 + 2y_1'}{y_1}dx} = e^{\int (p + \frac{2y_1'}{y_1})dx} = e^{[\int p(x)dx + \ln(y_1^2)]}$

$= \underline{y_1^2 \cdot e^{\int p(x)dx}}$

Hence $\frac{d}{dx}\left[y_1^2 e^{\int p(x)dx} u\right] = 0$

$v' = u = y_1^{-2} e^{-\int p(x)dx} K$

Therefore, $v(x) = \int u(x)dx = k\int y_1^{-2} e^{-\int p(x)dx} dx + C$; k, C constants

We may let $k = 1$, $C = 0$: $\boxed{y_2(x) = y_1(x) \cdot \int y_1^{-2} e^{-\int p(x)dx} dx}$

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2.2 (44)

$y_1(x) = x^{-\frac{1}{2}}\cos x$, $x > 0$ is a solution of Bessel's eq. of order $\frac{1}{2}$:

$L(y) = x^2 y'' + x y' + (x^2 - \frac{1}{4})y = 0$

[Order n: $x^2 y'' + xy' + (x^2 - n^2)y = 0$, p. 248: 3.5, EP]

Verification:

$y_1' = -\frac{1}{2}x^{-\frac{3}{2}}\cos x - x^{-\frac{1}{2}}\sin x$,

$y_1'' = \frac{3}{4}x^{-\frac{5}{2}}\cos x + \frac{1}{2}x^{-\frac{3}{2}}\sin x + \frac{1}{2}x^{-\frac{3}{2}}\sin x - x^{-\frac{1}{2}}\cos x$

$L(y_1) = (\frac{3}{4}x^{-\frac{1}{2}}\cos x + x^{\frac{1}{2}}\sin x - x^{\frac{3}{2}}\cos x)$

$+ (-\frac{1}{2})x^{-\frac{1}{2}}\cos x - x^{\frac{1}{2}}\sin x + (x^2 - \frac{1}{4})x^{-\frac{1}{2}}\cos x$

$= 0 \cdot x^{-\frac{1}{2}}\cos x + 0 \cdot x^{\frac{1}{2}}\sin x + 0\, x^{3/2}\cos x = 0$

So y_1 is a solution.

As in (36) we can use reduction of order to find an independent solution y_2: Had

$y_2(x) = y_1(x)\int y_1(x)^{-2} e^{-\int p(x)dx}dx$; $y_1(x) = x^{-\frac{1}{2}}\cos x$, $p(x) = x^{-1}$

$y_1^{-2} = x/\cos^2 x$

Hence

$y_2(x) = x^{-\frac{1}{2}}\cos x \int \frac{x}{\cos^2 x} e^{-\int \frac{1}{x}dx}dx = x^{-\frac{1}{2}}\cos x \underbrace{\int \frac{x}{\cos^2 x}\frac{1}{x}dx}_{\int \frac{1}{\cos^2 x}dx}$

$= x^{-\frac{1}{2}}\cos x \cdot (\tan x + k)$ (let $k = 0$)

$= \underline{x^{-\frac{1}{2}}\sin x}$

General solution of $L(y) = 0$: $\underline{y(x) = C_1 x^{-\frac{1}{2}}\cos x + C_2 x^{-\frac{1}{2}}\sin x}$ $(x > 0)$.

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2.3 (22) Solve:

$L(y) = 9y'' + 6y' + 4y = 0$, $y(0) = 3$, $y'(0) = 4$

Characteristic eq. $9r^2 + 6r + 4 = 0$, $108 = 27 \cdot 4 = 3 \cdot 9 \cdot 4$

$r = \frac{1}{18}[-6 \pm \sqrt{\underbrace{36 - 144}_{-108}}] = \frac{1}{18}[-6 \pm i6\sqrt{3}] = \frac{1}{3}[-1 \pm i\sqrt{3}]$

$= -\frac{1}{3} \pm i\frac{1}{\sqrt{3}}$

A general solution is

(*) $y(x) = e^{-\frac{1}{3}x}[A\cos(\frac{1}{\sqrt{3}}x) + B\sin(\frac{1}{\sqrt{3}}x)]$

$y(0) = 1 \cdot A = 3$

$y'(x) = -\frac{1}{3}y(x) + \frac{1}{\sqrt{3}}e^{-\frac{1}{3}x}[-A\sin(\frac{1}{\sqrt{3}}x) + B\cos(\frac{1}{\sqrt{3}}x)]$

$y'(0) = -\frac{1}{3}\cdot 3 + \frac{1}{\sqrt{3}}B = -1 + \frac{1}{\sqrt{3}}B = 4$, $B = \underline{5\sqrt{3}}$

The unique solution is

$\underline{y(x) = e^{\frac{1}{3}x}[3\cos(\frac{1}{\sqrt{3}}x) + 5\sqrt{3}\sin(\frac{1}{\sqrt{3}}x)]}$

2.3 (23)

$L(y) = y'' - 6y' + 25y = 0$, $y(0) = 3$, $y'(0) = +1$

Char. eq. $r^2 - 6r + 25 = 0$

$r = \frac{1}{2}[6 \pm \sqrt{\underbrace{36 - 100}_{-64}}] = \frac{1}{2}[6 \pm 8i] = 3 \pm 4i$

$y(x) = (C_1\cos 4x + C_2\sin 4x)e^{3x}$

$y(0) = C_1 \cdot 1 = 3$, $\underline{C_1 = 3}$.

$y'(x) = 3y(x) + 4e^{3x}(-C_1\sin(4x) + C_2\cos 4x)$

$y'(0) = 3 \cdot 3 + 4 \cdot C_2 = 1$, $C_2 = \frac{1}{4}\cdot(-8) = \underline{-2}$

$y(x) = \underline{(3\cos 4x - 2\sin 4x)e^{3x}}$.

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2.5 (1) A particular solution of

$L(y) = y'' + 16y = e^{3x}$

(char. roots $r = \pm 4i$, $L(y) = 0$: $y_c = A\cos 4x + B\sin 4x$)

We try

$y_p(x) = k e^{3x}$

Then

$L(y_p) = 9ke^{3x} + 16ke^{3x} = ke^{3x} \cdot 25 \equiv e^{3x}$

Hence $k = \frac{1}{25}$.

$y_p(x) = \frac{1}{25}e^{3x}$

(General sol. $y(x) = y_c(x) + y_p(x)$)

(3) $L(y) = y'' - y' - 3y = 2\sin 3x$

(char. roots are 3 and -2)

Try $y_p(x) = A\cos 3x + B\sin 3x$

$L(y_p) = -3(A\cos 3x + B\sin 3x) - \underbrace{3(-A\sin 3x + B\cos 3x)}_{y_p'} + \overbrace{(-9)(-A\cos 3x - B\sin 3x)}^{y_p''}$

$= (-15A - 3B)\cos 3x + (3A - 15B)\sin 3x = 2\sin 3x$, for all x

Hence

$15A + 3B = 0$, $\underline{B = -5A}$

$3A - 15B = 2$, $3A + 75A = 2$; $A = \frac{1}{39}$, $B = -\frac{5}{39}$

$y_p(x) = \underline{\frac{1}{39}\cos 3x - \frac{5}{39}\sin 3x}$

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2.5 (4)

$L(y) = \underline{4y''} + 4y' + y = 3x e^x$ $y' = e^x(A + B + Ax)$

(roots: $r = -\frac{1}{2}$, double root)

Try $y = y_p = (Ax + B)e^x$

$L(y_p) = e^x[Ax + B + 4(A + (Ax+B)) + \overbrace{4(A + B + Ax + A)}^{4y''}]$

$\equiv 3x e^x$, for all x

$\left.\begin{array}{l} 12A + 9B = 0 \\ 9A = 3 \end{array}\right\}$ $A = \frac{1}{3}$, $9B = -4$, $B = \frac{-4}{9}$

$y_p(x) = \underline{(\frac{1}{3}x - \frac{4}{9})e^x}$

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