Problem Set 5. Feb. 24. 2016.

FP: 2.5 (10). A particular solution of

$L(y) = y'' + 9y = \underline{2\cos 3x + 3\sin 3x}$

$L(y) = 0$ has characteristic roots

$r = \pm 3i$

Hence the complementary solutions are of the form

$y_c(x) = C_1 \cos 3x + C_2 \sin 3x$

Thus $2\cos 3x + 3\sin 3x$ is among the y_c-s. Therefore we should try

$y_p(x) = x \cdot (C_1 \cos 3x + C_2 \sin 3x) = x \cdot y_c(x)$

$y_p'(x) = 1 \cdot y_c(x) + x\, y_c'(x),$

$y_p''(x) = 2y_c'(x) + x\, y_c''(x)$

Hence

$L(y_p) = 2y_c'(x) + \underbrace{x\, y_c''(x) + 9x\, y_c(x)}$

$= 2y_c'(x) + x\, L(y_c) = 2y_c'(x)$

$= 2(-3C_1 \sin 3x + 3C_2 \cos 3x) \equiv 2\cos 3x + 3\sin 3x$ (for all x)

So $-6C_1 = \underline{3}$, $C_1 = -\frac{1}{2}$; $6C_2 = 2$, $\underline{C_2 = \frac{1}{3}}$

Therefore,

$y_p(x) = x\left(-\frac{1}{2}\cos 3x + \frac{1}{3}\sin 3x\right)$

is a particular solution.

feb 24-14:14

2.5(61)

$x^2y'' + xy' + y = \ln x$; $L(y) = \ln x$; $x > 0$

given that

$y_c(x) = C_1 \cos(\ln x) + C_2 \sin(\ln x)$; let $y_1(x) = \cos(\ln x)$, $y_2(x) = \sin(\ln x)$

solves $L(y) = 0$.

On "normal" form:

$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = \frac{\ln x}{x^2}$

Variation of the Parameters to find $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$

Then we obtain the system, linear in $u_1(x)$ and $u_2(x)$,

$$\left.\begin{array}{ll}(1) & u_1'y_1 + u_2'y_2 = 0 \\ (2) & u_1'y_1' + u_2'y_2' = \frac{\ln x}{x^2}\end{array}\right\} \text{ or } \begin{cases}\cos(\ln x)\,u_1' + \sin(\ln x)\,u_2' = 0 \\ -\frac{1}{x}\sin(\ln x)\,u_1' + \frac{1}{x}\cos(\ln x)\,u_2' = \frac{\ln x}{x^2}\end{cases}$$

From (1)

$u_1' = -\tan(\ln x)\, u_2'$

Using (2) this yields:

$-\tan(\ln x)\,u_2'\left(-\frac{1}{x}\sin(\ln x)\right) + \frac{1}{x}\cos(\ln x)\,u_2' = \frac{\ln x}{x^2}$

$\frac{1}{x}u_2'\left[\frac{+\sin^2(\ln x) + \cos^2(\ln x)}{\cos(\ln x)}\right] = \frac{1}{x\cos(\ln x)}u_2'$,

So $u_2' = \frac{\ln x}{x}\cos(\ln x)$ ($v = \ln x$, $dv = \frac{1}{x}dx$)

$u_2(x) = \int u_2'(x)dx = \int v\cos v\, dv = v\sin v - \int \sin v\, dv + C$ (let $C=0$)

$u_2(x) = \ln x \sin(\ln x) + \cos(\ln x)$

$u_1' = -\tan(\ln x)\,u_2' = -\frac{\ln x}{x}\sin(\ln x)$, $u_1(x) = \int u_1'(x)dx = -\int v \sin v\, dv = \dots$ ($v = \ln x$)

$= \ln x\cos(\ln x) - \sin(\ln x)$

$y_p(x) = u_1y_1 + u_2y_2 = \underline{\ln x\cos^2(\ln x)} - \sin(\ln x)\cos(\ln x) + \underline{\ln x\sin^2(\ln x)} + \sin(\ln x)\cos(\ln x)$

$= \underline{\ln x}$

SSS/SHSS

8.2.5 / (11.2.8)

feb 24-14:30

$\min_x \int_0^1 \underbrace{(x^2 + tx + tx\dot{x} + \dot{x}^2)}_{F(t,x,\dot{x})} dt$, $x(0) = 0$, $x(1) = 1$

Here

$\frac{\partial^2 F}{\partial x^2} = \underline{2 > 0}$, $\frac{\partial^2 F}{\partial \dot{x}^2} = \underline{2 > 0}$, $\frac{\partial^2 F}{\partial x \partial \dot{x}} = \frac{\partial}{\partial \dot{x}}(2x + t + t\dot{x} + 0) = t$

Hence

$$\Delta_F = \begin{vmatrix}\frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial \dot{x}} \\ \frac{\partial^2 F}{\partial \dot{x}\partial x} & \frac{\partial^2 F}{\partial \dot{x}^2}\end{vmatrix}_{(t,x,\dot{x})} = 2\cdot 2 - t^2 = 2^2 - t^2 > \underline{0} \text{ when } t \in [0,1].$$

Hence F is convex in $(x, \dot{x})$ for each fixed t in $[0,1]$.

Thus any solution x^* of the Euler eq. will solve the min. problem, provided that $x^*(0) = 0$ and $x^*(1) = 1$ ($x^* \in C^2[0,1]$)

The Euler eq. is:

$0 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 2x + t + t\dot{x} - \frac{d}{dt}(tx + 2\dot{x})$

$= 2x + t + \underline{t\dot{x}} - (1 \cdot x + \underline{t \cdot \dot{x}} + 2\ddot{x})$

$= -2\ddot{x} + x + t$

So

(E) $\underbrace{2\ddot{x} - x = t}$ (char. roots $r = \pm\frac{1}{\sqrt{2}}$)

$x_c(t) = C_1 e^{t/\sqrt{2}} + C_2 e^{-t/\sqrt{2}}$ solves $2\ddot{x} - x = 0$

Will find $x_p(t) = At + B$ (partic. solution)

$2\ddot{x}_p - x_p = 0 - At - B \equiv t$ for all $t \in [0,1]$

$-A = 1$, $B = 0$, so $x_p(t) = \underline{-t}$

General sol. $\underline{x(t) = C_1 e^{t/\sqrt{2}} + C_2 e^{-t/\sqrt{2}} - t}$

feb 24-14:52

Had

$x(t) = C_1 e^{t/\sqrt{2}} + C_2 e^{-t/\sqrt{2}} - t$

(the general solution of (E) $2\ddot{x} - x = t$)

Next, we must find $x = x^*$ such that

$0 = x^*(0) = C_1 + C_2$, $C_2 = -C_1$

and $1 = x^*(1) = C_1(e^{1/\sqrt{2}} - e^{-1/\sqrt{2}})$; $C_1 = \frac{1}{e^{1/\sqrt{2}} - e^{-1/\sqrt{2}}} = -C_2$

Thus $x^*(t) = 2\frac{e^{t/\sqrt{2}} - e^{-t/\sqrt{2}}}{e^{1/\sqrt{2}} - e^{-1/\sqrt{2}}} - t$ $\left(= 2\frac{\sinh(t/\sqrt{2})}{\sinh(1/\sqrt{2})}\right)$

is the unique C^2-solution of the min. problem.

ADVICE: Solve many of the problems before each problem session!

11.2.9 (SSS) Try instead

$\min \int_1^2 \underbrace{(x^2 + tx\dot{x} + t^2\dot{x}^2)}_{F(t,x,\dot{x})} dt$, $x(1) = 0$, $x(2) = 1$

Euler eq.: $0 = 2x + t\dot{x} - \frac{d}{dt}(tx + 2t^2\dot{x}) = 2x + t\dot{x} - (x + t\dot{x} + 4t\dot{x} + 2t^2\ddot{x})$

$\underbrace{0 = x - 4t\dot{x} - 2t^2\ddot{x}}$ or $\boxed{(E)\ \ \underbrace{2t^2\ddot{x} + 4t\dot{x} - x}_{L(x)} = 0}$

Try to find solutions $x = t^p$ (p a number):

$\dot{x} = pt^{p-1}$, $\ddot{x} = p(p-1)t^{p-2}$

$L(x) = t^p[2p(p-1) + 4p - 1] = t^p[2p^2 + 2p - 1] = 0$, all t

So $2p^2 + 2p - 1 = 0$, $p = \frac{1}{4}[-2 \pm \sqrt{4+8}] = \frac{1}{2}[-1 \pm \sqrt{3}] = r_1, r_2$

$x(t) = At^{r_1} + Bt^{r_2}$

feb 24-15:09

$x(t) = At^{r_1} + Bt^{r_2}$

where $0 = x(1) = A + B$, $B = -A$

and $1 = x(2) = A(2^{r_1} - 2^{r_2})$, $A = \frac{1}{2^{r_1} - 2^{r_2}} = -B$

$\underline{x(t) = \frac{t^{r_1} - t^{r_2}}{2^{r_1} - 2^{r_2}}}$ is the only possible sol.

May use 2nd derivative test to show F is convex in $(x, \dot{x})$ for each $t \in [1,2]$ (Check this!). Hence x is really a solution of the min. problem.

11.2.2 / 8.2.2

$\min \int_0^1 \underbrace{(tx + \dot{x}^2)}_{F(t,x,\dot{x})} dt$, $x(0) = 1$, $x(1) = 0$

(a) The Euler eq.:

$0 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 - \frac{d}{dt}(t + 2\dot{x}) = -(1 + 2\ddot{x})$

or $\underbrace{\ddot{x} = -\frac{1}{2}}$ ($\dot{x}(t) = -\frac{1}{2}t + C_1$)

Its solutions are

$\underbrace{x(t) = -\frac{1}{4}t^2 + C_1 t + C_2}$

(b) Here $x(0) = C_2 = 1$, $x(1) = -\frac{1}{4} + C_1 + 1 = 0$, $\underline{C_1 = -\frac{3}{4}}$

$\underline{x(t) = -\frac{1}{4}t^2 - \frac{3}{4}t + 1}$ (*)

is the unique such solution.

(c) Here $\frac{\partial^2 F}{\partial x^2} = 0 \geq 0$, $\frac{\partial^2 F}{\partial \dot{x}^2} = \frac{\partial}{\partial \dot{x}}(t + 2\dot{x}) = \underline{2} > 0$

$\frac{\partial^2 F}{\partial \dot{x}\partial x} = \frac{\partial}{\partial \dot{x}}(0) = \underline{0}$, $\Delta_F = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial \dot{x}^2} - \left(\frac{\partial^2 F}{\partial x \partial \dot{x}}\right)^2 = 0 \cdot 2 - 0 \geq 0$

Hence F is convex in $(x, \dot{x})$ for each t in $[0,1]$.

Therefore $x(t)$ in (*) is the unique C^2-solution of the min. problem.

feb 24-15:46