11.3.1 (SSS) / 8.3.1 (SHSS).

$$\min_x J(x) = \min_x \int_0^1 (x^2 + \dot{x}^2)\,dt \quad , \quad x(0) = 0,\ x(1) = 0$$

Here the integrand

$$F(t, x, \dot{x}) = x^2 + \dot{x}^2$$

is a convex function of the 2 variables $(x, \dot{x})$ (convex in $(x, \dot{x})$)
for each fixed t in $[0,1]$. (Cfr. the 2nd derivative test)
so we can solve the min problem by solving the Euler eq.

$$(E) \quad \ddot{x} - x = 0$$

$$x(t) = A e^t + B e^{-t}$$

$$x(0) = 0 \implies B = -A$$

$$x(1) = A(e + e^{-1}) = 0 \quad , \text{ so } \quad A = B = 0$$

$x(t) = 0$ (for all $t \in [0,1]$) is the only admissible solution of (E). Hence

$$J(0\cdot) = 0 = \min_x \int_0^1 (x^2 + \dot{x}^2)\,dt$$

The corresponding max problem does not have any solution

Let $\quad x(t) = \alpha(t - t^2), \quad x(t)^2 = \alpha^2(t^2 + t^4 - 2t^3)$

$$\dot{x}(t)^2 = \alpha^2(1 - 2t)^2 = \alpha^2(1 + 4t^2 - 4t)$$

$$\int_0^1 (x^2 + \dot{x}^2)\,dt = \alpha^2 \int_0^1 [1 - 4t + 5t^2 - 2t^3 + t^4]\,dt$$

$$= \alpha^2 \left[1 - 2 + \frac{5}{3} - \frac{1}{2} + \frac{1}{5} - 0\right] = \alpha^2 \frac{11}{30} \xrightarrow[\alpha \to \infty]{} \infty$$

Hence there can be no max. for $J(x)$.
We did not use the eq. (E) in this argument.

<u>Remark</u> Here $x(1) = x(0) = 0$, so the only C^2-function (admissible) that solves (E) is $x = 0$.
Since there are admissible x for which $J(x) > 0$, there can be no maximum x (of class C^2)

<u>11.3.2 / 8.3.3</u>

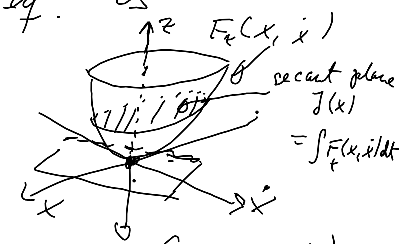
$$\min_x \int_a^1 t\dot{x}^2\, dt = \min_x J(x), \quad x(a) = 0, \ x(a) = 1$$

<u>Suppose $a \in (0,1)$:</u> $F(t, x, \dot{x}) = t\dot{x}^2$ is convex in $(x, \dot{x})$ for each t, since

$$\frac{\partial^2 F}{\partial x^2} = 0 \geq 0, \quad \frac{\partial^2 F}{\partial \dot{x}^2} = 2t > 0, \quad \frac{\partial^2 F}{\partial x \partial \dot{x}} = 0$$

$$\Delta_F = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial \dot{x}^2} - \left(\frac{\partial^2 F}{\partial x \partial \dot{x}}\right)^2 = 0 \geq 0$$

Hence F is convex. Here the Euler eq. is



$$0 - \frac{d}{dt}(2t\dot{x}) = -2[\dot{x} + t\ddot{x}] = 0$$

or

$$(E) \quad t\ddot{x} + \dot{x} = 0$$

<u>Easier:</u> $\frac{d}{dt}(2t\dot{x}) = 0$, $\quad 2t\dot{x} = C_1$ (constant)

$$(C = \tfrac{1}{2}C_1)$$

$$\dot{x} = \frac{C}{t}$$

$$x(t) = C \ln t + D \qquad (C, D \text{ arbitrary constants})$$

Here

$$x(1) = C \ln 1 + D = D = 1, \qquad \underline{D = 1}$$

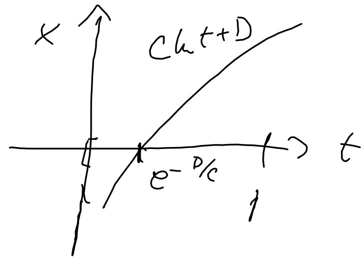
$$x(a) = C \ln a + 1 = 0, \qquad C = -\frac{1}{\ln a} \qquad (0 < a < 1)$$

$$x(t) = -\frac{\ln t}{\ln a} + 1 \qquad (a \in (0,1))$$

which solves the min problem (by convexity)

$\underline{a=0}$: Still have

(E) $\frac{d}{dt}(2t\dot{x}) = 0$



so $x(t) = C\ln t + D \quad 0 < t \leq 1$

would be the only possible solution candidate, but this x is not defined at $t = 0$, and is certainly not a C^2-function

($C = 0$ gives $x = D$, that is not admissible)

Hence no solution.

$\underline{11.5.2}$

(a) $\max_x J(x) = \max_x \int_0^1 \overbrace{(10 - \dot{x}^2 - 2x\dot{x} - 5x^2)e^{-t}}^{F(t,x,\dot{x})}\,dt$, $x(0)=0$, $x(1)=1$

$$\frac{\partial F}{\partial x} = (-2\dot{x} - 10x)e^{-t} = -2(\dot{x} + 5x)e^{-t}$$

$$\frac{\partial F}{\partial \dot{x}} = (-2\dot{x} - 2x - 0)e^{-t} = -2(\dot{x} + x)e^{-t}$$

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2(\ddot{x} + \dot{x})e^{-t} + 2(\dot{x} + x)e^{-t}$$
$$= -2e^{-t}[\ddot{x} - x]$$

Euler eq. $0 = [-\dot{x} - 5x + \ddot{x} - x]e^{-t}$

(E) $\ddot{x} - \dot{x} - 6x = 0$, char. roots: $r = \frac{1}{2}[1 \pm \sqrt{25}] = \begin{cases} 3 \\ -2 \end{cases}$

General solution:

$$x(t) = Ae^{3t} + Be^{-2t}$$

$$x(0) = A + B = 0, \quad B = -A$$

$$x(1) = Ae^3 + (-A)e^{-2} = A(e^3 - e^{-2}) = 1,$$

$$A = \frac{1}{e^3 - e^{-2}} = -B$$

The unique admissible solution:

$$(*) \quad x(t) = \frac{1}{e^3 - e^{-2}}\left(e^{3t} - e^{-2t}\right)$$

Here $\dfrac{\partial^2 F}{\partial x^2} = -10e^{-t} < 0$, $\dfrac{\partial^2 F}{\partial \dot{x}^2} = -2e^{-t} < 0$,

$$\frac{\partial^2 F}{\partial x \partial \dot{x}} = -2e^{-t}$$

$$\Delta_F = 20e^{-2t} - 4e^{-2t} = 16e^{-2t} > 0$$

Hence F is (strictly) concave in $(x, \dot{x})$ for each t.

Hence $(*)$ solves the max problem.

(b) Suppose next that $x(0) = 0$ and $x(1)$ is free.
The Euler eq. is as before, so $x(0)=0$ yields $B=-A$,

$$x(t) = A\left(e^{3t} - e^{-2t}\right)$$

By the Transversality cond. (T), we must have

$$0 = \left(\frac{\partial F}{\partial \dot{x}}\right)_{t=1} = -2(\dot{x}+x)e^{-t}\Big|_{t=1} = -2\left(\dot{x}(1)+x(1)\right)e^{-1}$$

So $\dot{x}(1) + x(1) = 0$

$\Updownarrow$ $A\left(3e^{3t} + 2e^{-2t} + e^{3t} - e^{-2t}\right)_{t=1} = 0$

$\Updownarrow$ $A\left(4e^{3} + e^{-2}\right) = 0$

$\Updownarrow$ $\underline{A = 0}$

Hence $x = 0$ is the unique solution:

$\underline{J(0) = \max_x J(x)}$ $\quad \left(J(0) = \int_0^1 10e^{-t}dt = \Big|_0^1 -10e^{-t} = -10e^{-1} + 10 = \underline{10(1-\tfrac{1}{e})}\right)$

Exam June 2006 (1)

$\min_x \int_1^2 \left(\frac{x^2}{t^2} + \frac{\dot{x}^2}{2}\right) dt$, $x(1)=1$, $x(2)$ free

(a) The Euler eq. for $F(t,x,\dot{x}) = \frac{x^2}{t^2} + \frac{\dot{x}^2}{2}$. is

$$0 = \frac{2x}{t^2} - \frac{d}{dt}(\dot{x}) = \frac{2x}{t^2} - \ddot{x}$$

$$(E) \quad \ddot{x} - \frac{2}{t^2}x = 0$$

Look for solutions $x(t) = t^p$: $\dot{x}(t) = pt^{p-1}$

$\ddot{x}(t) = p(p-1)t^{p-2}$

Into (E) :

$t^{p-2}[p(p-1) - 2] = t^{p-2}[p^2 - p - 2] = 0$, $t \in [1, 2]$

$\Leftrightarrow \quad p^2 - p - 2 = 0 \Leftrightarrow p = \frac{1}{2}[1 \pm \sqrt{1+8}] = \begin{cases} 2 \\ -1 \end{cases}$

So 2 linearly indep. solutions are t^2 and t^{-1} on $[1,2]$. ($t^2/t^{-1} = t^3$ non constant)

(b) The general solution of (E) is

$x(t) = At^2 + Bt^{-1}$ (A, B arbitrary constants)

Here $x(1) = A + B = 1$, $B = 1 - A$

$x(2)$ is free, so

$0 = \left(\frac{\partial F}{\partial \dot{x}}\right)_{t=2} = \dot{x}(2)$,

So $(2At - Bt^{-2})_{t=2} = 4A - B\frac{1}{4} = 4A - (1-A)\frac{1}{4} = 0$

$4A + \frac{1}{4}A = 1/4$, $\frac{17}{4}A = 1/4$, $\underline{A = \frac{1}{17}}$

$B = 1 - A = \frac{16}{17}$, $\underline{x(t) = \frac{1}{17}t^2 + \frac{16}{17}t^{-1}}$

is the only solution candidate.

Here $F(t, x, \dot{x})$ is the sum of 2 convex functions each of 1 variable $f_t(x) = \frac{x^2}{t}$ and $g(\dot{x}) = \frac{1}{2}\dot{x}^2$, hence
$F(t,x,\dot{x}) = f_t(x) + g(\dot{x})$ is convex in $(x, \dot{x})$ for each t in $[1,2]$. Hence (by Thm. 2) $x(t) = \frac{1}{17}t^2 + \frac{16}{17}t^{-1}$ solves the min. problem.

Alternative: 2nd derivative test yields that F is (strictly) convex in $(x, \dot{x})$

$$\frac{\partial^2 F}{\partial x^2} = \frac{2}{t^2} > 0, \quad \frac{\partial^2 F}{\partial \dot{x}^2} = 1 > 0, \quad \frac{\partial^2 F}{\partial x \partial \dot{x}} = 0$$

$$\Delta_F = \begin{vmatrix} \frac{\partial^2 F}{\partial x^2} & \frac{\partial^2 F}{\partial x \partial \dot{x}} \\ \frac{\partial^2 F}{\partial x \partial \dot{x}} & \frac{\partial^2 F}{\partial \dot{x}^2} \end{vmatrix} = \frac{2}{t^2} - 0 > \underline{0}$$

Extra: 11.3.3 with $T = 4$:

$$\min_x \int_0^4 (\dot{x}^2 - x^2)\,dt, \quad x(0) = x(4) = 0$$

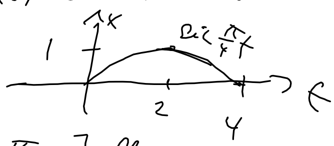
The Euler eq. is

$$0 = \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = -2x - \frac{d}{dt}(2\dot{x}) = -2x - 2\ddot{x}$$

or (E) $\ddot{x} + x = 0$

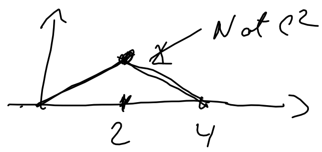
Char. roots $r^2 + 1 = 0$, $r = \pm i$,
$x(t) = A\cos t + B\sin t$, $x(0) = A = \underline{0}$, $x(4) = B\sin 4 = 0$
$B = 0$, $x = 0$ is only solution on $[0,4]$ that is admissible. The 2nd derivative test shows F is neither convex nor concave in $(x, \dot{x})$ here. (check this!). Let instead

$x(t) = \sin\left(\frac{\pi}{4}t\right)$. Clearly, $x \in C^2[0,4]$ and $x(0) = 0 = x(4) = 0$



Furthermore, $\dot{x}(t) = \frac{\pi}{4}\cos\frac{\pi}{4}t$

$$J(x) = \int_0^4 (\dot{x}^2 - x^2)dt = \int_0^4 \left[\frac{\pi^2}{16}\cos^2\frac{\pi}{4}t - \sin^2\frac{\pi}{4}t\right]dt$$



$$= \int_0^4 \left[\frac{\pi^2}{16} - \frac{\pi^2}{16}\sin^2\frac{\pi}{4}t - \sin^2\frac{\pi}{4}t\right]dt$$

$$= \int_0^4 \left[\frac{\pi^2}{16} - \left(\frac{\pi^2}{16}+1\right)\sin^2\left(\frac{\pi}{4}t\right)\right]dt$$

$$= \frac{\pi^2}{16}\cdot 4 - \left(\frac{\pi^2}{16}+1\right)\int_0^4 \frac{1}{2}\left(1 - \cos\frac{\pi}{2}t\right)dt = \frac{\pi^2}{4} - \left(\frac{\pi^2}{16}+1\right)\frac{1}{2}\Big|_0^4\left(t - \frac{2}{\pi}\sin\frac{\pi}{2}t\right)$$

$$= \frac{\pi^2}{8} - 2 < 0$$

However, with $x = x^* = 0$ (the only admissible solution of (E)), we find

$$J(x^*) = J(0) = \int_0^4 [(\dot{x}^*)^2 - x^{*2}]dt = 0 \geq 0.$$

Hence x^* cannot give a minimum. So there is no C^2-solution to the min problem.