# SET 7 (March 9, 2016) MAT 2440

## EP 5.1 (8)

(1) $x'' + 3x' + 4x - 2y = 0$

(2) $y'' + 2y' - 3x + y = \cos t$

To transform into 1st order linear system:
Let

$$x_1 = x, \quad x_2 = x_1' = x' \quad (\text{then } x_2' = x'')$$

$$y_1 = y, \quad y_2 = y_1' = y' \quad (\text{so } y_2' = y'')$$

Get:

$$\begin{cases} x_1' = x_2 \\ x_2' + 3\underbrace{x_1'}_{x_2} + 4x_1 - 2y_1 = 0 \\ y_1' = y_2 \\ y_2' + 2\underbrace{y_1'}_{y_2} - 3x_1 + y_1 = \cos t \end{cases}$$

or

$$\begin{cases} x_1' = x_2 \\ x_2' = -4x_1 - 3x_2 + 2y_1 \\ y_1' = y_2 \\ y_2' = 3x_1 - y_1 - 2y_2 + \cos t \end{cases}$$

or on matrix form

$$\begin{bmatrix} x_1' \\ x_2' \\ y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -4 & -3 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cos t \end{bmatrix}$$

Not required for the problem:

Eigenvalues of matrix

$|\lambda I - P| = 0$, get $\lambda^4 + 5\lambda^3 + 11\lambda^2 + 5\lambda + 4 = 0$

MATLAB: $A = [0,1,0,0; -4,-3,2,0; \cdots]$

$[V, D] = eig(A)$

Get

$\lambda = -0.2911 \pm i\,0.4931\,;\; -2.2089 \pm i\,1.1049$

$(\lambda_1, \ldots, \lambda_4)$

Control:

$\prod_{k=1}^{4} \lambda_k = 4\,, \quad \sum_{k=1}^{4} \lambda_k = -5$ Ok

Eigenvectors:

$$\vec{v}_1 = \begin{bmatrix} -0.4269 + i\,0.1715 \\ 0.0397 - i\,0.2604 \\ -0.7358 \\ 0.2142 - i\,0.3628 \end{bmatrix}, \quad \vec{v}_2 = \cdots, \quad \vec{v}_4 = \cdots$$

5.1 (21) a (Cfr. 5.1.11)

(i) $x' = y$

(ii) $y' = -x$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

By elimination: $x'' + x = 0$ : $r = \pm i$,

$x(t) = A\cos t + B\sin t$

$y(t) = -A\sin t + B\cos t$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = A \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + B \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Hence $\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$, $\begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$

are linearly independent vector solutions. They generate the (2-dimensional) solution space.

Here

$$x(t)^2 + y(t)^2$$
$$= A^2\cos^2 t + B^2\sin^2 t + 2AB\sin t\cos t$$
$$+ A^2\sin^2 t + B^2\cos^2 t - 2AB\sin t\cos t$$
$$= A^2 \cdot 1 + B^2 \cdot 1 = r^2 \geq 0$$

So solution curves are circles centered at (0,0) with radius $r \geq 0$, $r = \sqrt{A^2+B^2}$

(b) Consider

$$x' = y, \quad y' = x \qquad \text{or} \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Elimination:

$$x'' = y' = x, \qquad x'' - x = 0$$

$r = \pm 1$,

$$x(t) = Ae^{-t} + Be^{t}$$

$$y(t) = x'(t) = -Ae^{-t} + Be^{t}$$

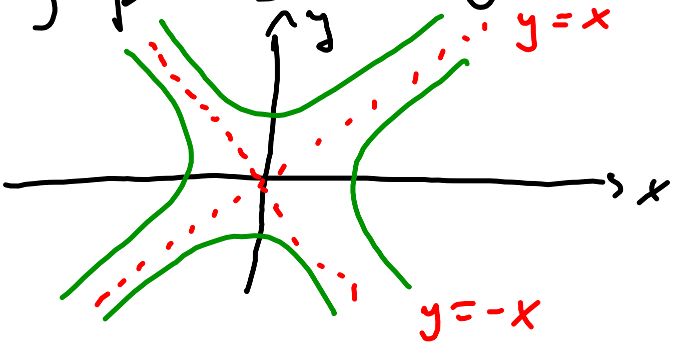
$$x(t)^2 - y(t)^2 = A^2e^{-2t} + B^2e^{2t} + 2AB - (A^2e^{-2t} + B^2e^{2t} - 2AB)$$

$$= 4AB = C = \text{constant.}$$

Solution curves:

$$(x - y)(x + y) = C,$$

hyperbolas centered at $(0,0)$, with asymptotes: $y = x$ and $y = -x$



5.3 (21) Given

$$(*) \quad \vec{x}\,' = \underbrace{\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix}}_{A} \vec{x} \quad , \quad \text{or} \quad \vec{x}\,' = A\vec{x}$$

Then $\vec{x}_1(t) = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix}$ and $\vec{x}_2(t) = \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix}$

are solutions of $(*)$:

$$\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \vec{x}_1(t) \quad ,$$

and

$$\vec{x}_1\,'(t) = \begin{bmatrix} 2e^t \\ -3e^t \end{bmatrix} = \vec{x}_1(t)$$

so $\vec{x}_1\,' = A\vec{x}_1 = \vec{x}_1$ ; $\vec{x}_1$ eigenvector with $\lambda = 1$ as <u>eigenvalue</u>

Furthermore,

$$\begin{bmatrix} 4 & 2 \\ -3 & -1 \end{bmatrix} \begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ -2e^{t+2t} \end{bmatrix} = 2\begin{bmatrix} e^{2t} \\ -e^{2t} \end{bmatrix} = 2\vec{x}_2(t)$$

and

$$\vec{x}_2\,'(t) = \begin{bmatrix} 2e^{2t} \\ -2e^{2t} \end{bmatrix} = 2\vec{x}_2(t)$$

so $\vec{x}_2$ is a solution. Observe: $\vec{x}_2$ is eigenvector with eigenvalue 2 for $A$. Solution space is <u>span $\{\vec{x}_1, \vec{x}_2\}$</u>

<u>5.4</u> (17) "Eigenvalue method":

$$\left.\begin{aligned} x_1' &= 4x_1 + x_2 + 4x_3 \\ x_2' &= x_1 + 7x_2 + x_3 \\ x_3' &= 4x_1 + x_2 + 4x_3 \end{aligned}\right\} \quad \underbrace{\begin{bmatrix} 4 & 1 & 4 \\ 1 & 7 & 1 \\ 4 & 1 & 4 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}}_{\vec{x}'}$$

or $\vec{x}' = P\vec{x}$

<u>Notice</u> that $\lambda = 0$ is an eigenvalue of $P$, as 1st and 3rd rows are equal (so $\det(0 \cdot I - P) = \det P = 0$)

Characteristic equation of $P$:

$$0 = |\lambda I - P| = \begin{vmatrix} \lambda-4 & -1 & -4 \\ -1 & \lambda-7 & -1 \\ -4 & -1 & \lambda-4 \end{vmatrix} \quad (R_3 \to R_3 - 4R_2)$$

$$= \begin{vmatrix} \lambda-4 & -1 & -4 \\ -1 & \lambda-7 & -1 \\ 0 & 27-4\lambda & \lambda \end{vmatrix} \quad (R_1 \to R_1 + (\lambda-4)R_2$$

$$= \begin{vmatrix} 0 & (\lambda-4)(\lambda-7)-1 & -\lambda \\ -1 & \lambda-7 & -1 \\ 0 & 27-4\lambda & \lambda \end{vmatrix}$$

$$= 1 \cdot [[(\lambda-4)(\lambda-7)-1]\lambda + (27-4\lambda)\lambda]$$

$$= \lambda[\lambda^2 - 11\lambda + 27 + 27 - 4\lambda]$$

$$= \lambda(\lambda^2 - 15\lambda + 54)$$

$$(\lambda = \tfrac{1}{2}[15 \pm \sqrt{\underbrace{225-216}_{9}}] = \begin{cases} 6 \\ 9 \end{cases}, 0)$$

# Eigenvectors

**$\lambda = 0$**  $\lambda I - P \sim P \sim \begin{bmatrix} 1 & 7 & 1 \\ 4 & 1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 1 \\ 0 & -27 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,  $P\vec{x} = \vec{0}$ , $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\left.\begin{matrix} x_1 + x_3 = 0 \\ x_2 = 0 \end{matrix}\right\} \begin{matrix} x_3 = -x_1 \\ x_2 = 0 \end{matrix}$

Eigenspace spanned by $\underline{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \vec{v}_1}$

**$\lambda = 6$:**  $6I - P \sim \begin{bmatrix} 0 & 2(-1)-1 & -6 \\ -1 & -1 & -1 \\ 0 & 3 & 6 \end{bmatrix}$

$\sim \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\underline{x_1 = x_3}$ , $x_2 + 2x_3 = 0$

$\underline{x_2 = -2x_1}$

Eigenspace spanned by $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \vec{v}_2$

Since $P$ is symmetric, $P^t = P$, eigenspaces are pairwise orthogonal. Let $\vec{v}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be eigenvector for $\lambda = 9$:

$0 = \vec{v}_1 \cdot \vec{v}_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = x_1 - x_3$ , $\underline{x_3 = x_1}$

and $0 = \vec{v}_2 \cdot \vec{v}_3 = x_1 - 2x_2 + x_3 = 2(x_1 - x_2)$

$\underline{x_1 = x_2}$  $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

A general sol. for system is

$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{6t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{9t}$$

$$= \begin{bmatrix} c_1 + c_2 e^{6t} + c_3 e^{9t} \\ -2c_2 e^{6t} + c_3 e^{9t} \\ -c_1 + c_2 e^{6t} + c_3 e^{9t} \end{bmatrix}$$

($c_1, c_2, c_3$ are arbitrary constants)

<u>Extra</u>: Find $F(t, x, \dot{x})$ with Euler equation $\ddot{x} - x = 0$

$\left( \frac{\partial F}{\partial x} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = 0 \right)$

Try $\frac{\partial F}{\partial \dot{x}} = \dot{x}$, so $\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{x}}\right) = \ddot{x}$; $\frac{\partial F}{\partial x} = -x$

so $F(t, x, \dot{x}) = -\frac{1}{2}x^2 + G(t, \dot{x})$

Hence we try $\underline{F(t, x, \dot{x}) = +\frac{1}{2}x^2 + \frac{1}{2}\dot{x}^2}$

More general: $F(t, x, \dot{x}) = x^2 + \dot{x}^2 + g(x)\dot{x} + f(t)$

## Simpson's Rule: (Proof: Any good book in Calculus)

$$\int_a^{a+h} f(t)\,dt \qquad (f \in C^4[a, a+h])$$

$$= \frac{h}{6}\left[f'(a) + 4f'(a+\tfrac{h}{2}) + f'(a+h)\right] + Er$$

where $Er = -\frac{h^5}{2880} f^{(4)}(c)$, some $c \in (a, a+h)$

Let's try it on

$$\max_x Si(x) = Si(\pi) = \int_0^{\pi} \frac{\sin t}{t}\,dt$$

$Si'(0) = 1$, $\quad Si'(\pi) = \frac{\sin \pi}{\pi} = 0$

$Si'(\frac{\pi}{2}) = \frac{\sin\frac{\pi}{2}}{\pi/2} = \frac{2}{\pi}$

So

$$Si(\pi) \approx \frac{\pi}{6}\left[1 + 4\cdot\frac{2}{\pi} + 0\right]$$

$$= \frac{\pi}{6}\left(1 + \frac{8}{\pi}\right) = 1.8569\ldots\ldots$$

KNOWN: $Si(\pi) = 1.85193705\ldots\ldots$

So $Er = 1.8519 - 1.8569 = -0.005\ldots = -5\cdot 10^{-3}$

Problem (a) Do this for 2 subintervals: $[0, \frac{\pi}{2}]$ and $[\frac{\pi}{2}, \pi)$

(b) for 4 subintervals: $[0, \frac{\pi}{4}], \ldots, [\frac{3\pi}{4}, \pi]$

(c) Read about Euler's method and the improved Euler method: EP p. 430 Ch. 6.1