We considered linear systems

(*) $\vec{x}\,' = P\vec{x}$

Had discussed the case of complex eigenvalues λ of P

<u>Found:</u> If $\vec{x}$ is a solution of (*), then also $\vec{x}_1 = \mathrm{Re}\,\vec{x}$ and $\vec{x}_2 = \mathrm{Im}\,\vec{x}$ are solutions.

Moreover, need not consider $\bar{\lambda}$ and $\vec{\bar{x}}$ associated to eigenvalue $\lambda = p + iq$, $q, p \in \mathbb{R}$

If $\vec{v} = \vec{a} + i\vec{b}$, $\vec{a}, \vec{b}$ real vectors is a corresp. eigenvector

We may write a solution

$$\vec{x}(t) = e^{\lambda t}\vec{v}$$

(of (*)). Then

$$\vec{x}(t) = e^{pt}(\cos qt + i \sin qt)(\vec{a} + i\vec{b})$$
$$= e^{pt}\left((\cos qt\,\vec{a} - \sin qt\,\vec{b}) + i(\sin qt\,\vec{a} + \cos qt\,\vec{b})\right)$$

Hence

$$(**)\begin{cases} \vec{X}_1(t) = e^{pt}(\vec{a}\cos qt - \vec{b}\sin qt) = \operatorname{Re}\vec{X}(t) \\ \vec{X}_2(t) = e^{pt}(\vec{a}\sin qt + \vec{b}\cos qt) = \operatorname{Im}\vec{X}(t) \end{cases}$$

are real valued solutions of $(*)$ associated to λ and $\bar{\lambda}$

<u>They are linearly independent</u>:

Write $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$, $\vec{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$.

Suppose

$$\alpha \vec{X}_1(t) + \beta \vec{X}_2(t) \equiv 0$$

Then, for each $k = 1, 2, \dots, n$, we see from $(**)$ that

$$\left.\begin{aligned} &\alpha \cancel{e^{pt}}(a_k \cos qt - b_k \sin qt) \\ &+\beta \cancel{e^{pt}}(a_k \sin qt + b_k \cos qt) \equiv 0 \end{aligned}\right\} (i).$$

Differentiating (i):

$$\left.\begin{aligned} &\alpha(-a_k \sin qt - b_k \cos qt) \\ &+\beta(a_k \cos qt - b_k \sin qt) \equiv 0 \end{aligned}\right\} (ii)$$

(i) & (ii) form a linear system in the unknown α and β

The system has determinant

$a_k^2 + b_k^2 \qquad (1 \leq k \leq n)$

(Check this!)

hence the determinant $a_k^2+b_k^2 \neq 0$ for at least one k (otherwise $\vec{v} = \vec{0}$, impossible).

For such k we find $\alpha = \beta = 0$ is the unique solution.

Hence $\vec{X}_1$ and $\vec{X}_2$ are linearly independent.

Problem Set 8. March 30. 2016.

5.4 (24) On matrix form:

$$\vec{X}' = \begin{bmatrix} 2 & 1 & -1 \\ -4 & -3 & -1 \\ 4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = P\vec{X}.$$

Eigenvalues of P:

$$0 = |\lambda I - P| = \begin{vmatrix} \lambda-2 & -1 & 1 \\ 4 & \lambda+3 & 1 \\ -4 & -4 & \lambda-2 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda-2 & -1 & 1 \\ 4 & \lambda+3 & 1 \\ 0 & \lambda-1 & \lambda-1 \end{vmatrix} \quad (R_3 \mapsto R_3 + R_2)$$

$$= (\lambda-1)\begin{vmatrix} \lambda-2 & -1 & 1 \\ 4 & \lambda+3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = (\lambda-1)\begin{vmatrix} \lambda-2 & -2 & 0 \\ 4 & \lambda+2 & 0 \\ 0 & 1 & 1 \end{vmatrix} \quad \begin{matrix} R_1 \mapsto R_1 - R_3 \\ R_2 \mapsto R_2 - R_3 \end{matrix}$$

$$= (\lambda-1)(\lambda^2 - 4 + 8) = (\lambda-1)(\lambda^2+4)$$

<u>Eigenvalues</u>: $\lambda = 1$, $\lambda = \pm 2i$

Eigenspaces are one-dimensional here.

$\underline{\lambda = 1}$ ~~Take~~ $\vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ eigenvector:

$$I - P \sim \begin{bmatrix} -1 & -1 & 1 \\ 4 & 4 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_3 = 0$, $x_1 + x_2 = 0$; let $\vec{v}_1 = \underline{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$

$\underline{\lambda = 2i}$

$$2iI - P \sim \begin{bmatrix} 2i-2 & -2 & 0 \\ 4 & 2i+2 & 0 \\ 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} i-1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$\vec{w}_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ eigenvector: $x_2 + x_3 = 0$, $x_3 = -x_2$

$(i-1)x_1 = x_2$

Let $x_1 = 1$, $x_2 = (i-1)$, $x_3 = 1 - i$

$$\vec{w}_2 = \begin{bmatrix} 1 \\ i-1 \\ 1-i \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Let $\underline{\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}$

The general solution of ~~our~~ system is:

$$\vec{X}(t) = c_1 e^t \vec{v}_1 + c_2 \vec{X}_2(t) + c_3 \vec{X}_3(t)$$

$$\vec{X}_2(t) = \begin{bmatrix} \cos 2t \\ -\sin 2t - \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix}$$

$$\vec{X}_3(t) = \begin{bmatrix} \sin 2t \\ \cos 2t - \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$$

This can be seen from

$$\begin{bmatrix} 1 \\ i-1 \\ 1-i \end{bmatrix} e^{i2t} = \begin{bmatrix} 1 \\ i-1 \\ 1-i \end{bmatrix} (\cos 2t + i \sin 2t)$$

$$= \begin{bmatrix} \cos 2t \\ -\sin 2t - \cos 2t \\ \cos 2t + \sin 2t \end{bmatrix} + i \begin{bmatrix} \sin 2t \\ \cos 2t - \sin 2t \\ \sin 2t - \cos 2t \end{bmatrix}$$

<u>5.4.20</u> On matrix form:

$$\vec{X}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 5 & 1 & 3 \\ 1 & 7 & 1 \\ 3 & 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \; ; \quad \vec{X}' = P\vec{X}$$

Here P is symmetric, $P = P^T$, hence eigenspaces are pairwise orthogonal, relatively to the standard inner product on $\mathbb{R}^3$.

We make row reductions on

$$\lambda \underline{I} - P = \begin{bmatrix} \lambda-5 & -1 & -3 \\ -1 & \lambda-7 & -1 \\ -3 & -1 & \lambda-5 \end{bmatrix} \sim \begin{bmatrix} \lambda-5 & -1 & -3 \\ 1 & 7-\lambda & 1 \\ 0 & 3\lambda-20 & 2-\lambda \end{bmatrix}$$

$(R_3 \mapsto R_3 - 3R_2)$
$(R_2 \mapsto -R_2)$

$$\sim \begin{bmatrix} 0 & \lambda^2-12\lambda+34 & 2-\lambda \\ 1 & 7-\lambda & 1 \\ 0 & 3\lambda-20 & 2-\lambda \end{bmatrix}$$

$R_1 \mapsto R_1 - (\lambda-5)R_2$

$$|\lambda I - P| = (-1)\left[(\lambda^2-12\lambda+34)(2-\lambda) - (2-\lambda)(3\lambda-20)\right]$$

$$= (-1)(2-\lambda)(\lambda^2-15\lambda+54)$$

Eigenvalues: $\underline{\lambda = 2}$,

$$\lambda = \frac{1}{2}\left[15 \pm \sqrt{225-216}\right] = \frac{1}{2}[15 \pm 3] = \begin{cases} 9 \\ \underline{6} \end{cases}$$

We had:

$$\lambda I - P \sim \begin{bmatrix} 1 & 7-\lambda & 1 \\ 0 & \lambda^2-12\lambda+34 & 2-\lambda \\ 0 & 3\lambda-20 & 2-\lambda \end{bmatrix}$$

$\underline{\lambda = 2}$: $2I - P \sim \begin{bmatrix} 1 & 5 & 1 \\ 0 & 14 & 0 \\ 0 & -14 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 5 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\quad \vec{v}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ eigenvector

$x_2 = 0, \quad x_3 = -x_1$

Take $\vec{v}_1 = \underline{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}$

$\underline{\lambda = 6:}$

$$6I - P \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_2 := -2x_3$, $x_1 + x_2 + x_3 = 0$

So $x_1 - x_3 = 0$, $x_3 = x_1$,

$x_2 = -2x_1$

Get basisvector for eigenspace:

$$\underline{\vec{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

$\underline{\lambda = 9:}$ $9I - P \sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 7 & -7 \\ 0 & 7 & -7 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$: $x_3 = x_2$

$x_1 - 2x_2 + x_2 = 0$

So $x_1 = x_2$, $x_3 = x_2$

$$\underline{\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

Check: They are pairwise orthogonal.

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, so a general solution is

$$\vec{x}(t) = \underline{c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{6t} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + c_3 e^{9t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$$

c_1, c_2, c_3 arbitrary constants.

5.6.1 A general solution of

$$\vec{x}' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix} \vec{x} \quad = P\vec{x}$$

Here

$$|\lambda I - P| = \begin{vmatrix} \lambda+2 & -1 \\ 1 & \lambda+4 \end{vmatrix} = (\lambda+2)(\lambda+4)+1$$

$$= \lambda^2 + 6\lambda + 9 = 0$$

$\Leftrightarrow \ \underline{\lambda = -3}$: The only eigenvalue.

Eigenspace:

$$-3I - P = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$x_2 = -x_1$; $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

is a basis vector for the one-dim. eigenspace. Hence the eigenvalue <u>method fails here</u>.

Instead we use elimination:

(1) $x_1' = -2x_1 + x_2$ $\quad | \ x_2 = x_1' + 2x_1$

(2) $x_2' = -x_1 - 4x_2$

We differentiate (1) and use (2), and finally use (1):

$$x_1'' = -2x_1' + x_2'$$

$$\overset{(2)}{=} -2x_1' - x_1 - 4x_2$$

$$\overset{(1)}{=} -2x_1' - x_1 - 4(x_1' + 2x_1)$$

$$= -6x_1' - 9x_1 .$$

Hence

$$x_1'' + 6x_1' + 9x_1 = 0$$

with char. equation

$$r^2+6r+9=0$$

$$r = \frac{1}{2}\left[-6 \pm \sqrt{36-36}\right] = \underline{-3}$$

Hence

$$\underline{x_1(t) = c_1 e^{-3t} + c_2 t e^{-3t}}$$

Then

$$x_2 = 2x_1 + x_1' = 2c_1 e^{-3t} + 2c_2 t e^{-3t}$$

$$+ (-3) c_1 e^{-3t} + c_2 e^{-3t} - 3c_2 t e^{-3t}$$

$$= -c_1 e^{-3t} + c_2 e^{-3t} - c_2 t e^{-3t}$$

$$= -c_1 e^{-3t} + c_2 e^{-3t}(1-t)$$

Then

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underline{c_1 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-3t}\left(t\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)}$$

Let $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

We notice that

$$[P - (-3)I]\vec{v}_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{v}_1$$

and hence

$$(P-(-3)I)^2\vec{v}_2 = (P-(-3)I)\vec{v}_1 = \vec{0},$$

since $\vec{v}_1$ was an eigenvector for P.

<u>This indicates a possible way of improving the eigenvalue method in such cases.</u>